GEOMETRIC SEQUENCES AND SERIES

OUTCOMES:

1. Recognise a geometric sequence.
2. Find the general term of a geometric sequence.
3. Use the general term formula to determine the position of a term and finding a term in a given position.
4. Find the sum of a geometric series.

INTRODUCTION:

Explore the following number pattern.

$$2;\ \ 8;\ \ 32;\ \ 128;\ \dots\dots$$

You will agree that each term is **multiplied** by a certain constant. This value is known as the common ratio (r).

The constant can be determined by:

$$r = \frac{T_2}{T_1} = \frac{T_3}{T_2}$$

PLEASE NOTE: a is the first term and r is the common ratio, therefore:

$$a = 2 \quad \text{and} \quad r = 4$$

Lets explore this further:

$$2;\ \ 8;\ \ 32;\ \ 128;\ \dots\dots$$

$$T_1 = 2 = a$$
$$T_2 = 2 \times 4 = a \times r = ar$$
$$T_3 = 2 \times 4 \times 4 = a \times r \times r = ar^2$$
$$T_4 = 2 \times 4 \times 4 \times 4 = a \times r \times r \times r = ar^3$$
$$T_5 = ar^{5-1} = ar^4$$
$$\therefore T_{10} = ar^9$$

This formula can be used to determine any term in this type of sequence.

$$T_n = ar^{n-1}$$

T_n is the general term or the last term

WORKED EXAMPLE 1:

Write down the value of a and r in each of the following cases

a) $1;\ -2;\ \ 4;\ -8;\ \ldots..$

$a = 1$	$r = \frac{T_2}{T_1} = -\frac{2}{1} = -2$

b) $2;\ -1;\ \frac{1}{2};\ -\frac{1}{4};\ \ldots.$

$a = 2$	$r = \frac{T_2}{T_1} = -\frac{1}{2}$

c) $243;\ \ 81;\ \ 27;\ \ldots..$

$a = 243$	$r = \frac{T_2}{T_1} = \frac{81}{243} = \frac{1}{3}$

WORKED EXAMPLE 2:

Determine the general term of the following sequences.

STEP 1: Find the value of a (First term).

STEP 2: Determine the value of r (Common ratio)

STEP 3: Substitute into the T_n formula

STEP 4: Simplify

a) $2;\ \ 8;\ \ 32;\ \ 128;\ \ldots\ldots$

STEP 1: Find the value of a (First term)

$a = 2$

STEP 2: Determine the value of r (Common ratio)

$$r = \frac{T_2}{T_1} = \frac{8}{2} = 4$$

	STEP 3: Substitute into the T_n formula $T_n = ar^{n-1}$ $T_n = 2(4)^{n-1}$ STEP 4: Simplify $T_n = 2(4)^{n-1}$; $T_1 = 2(4)^{1-1}$; $T_1 = 2$ | $T_n = 2(4)^{n-1}$; $T_2 = 2(4)^{2-1}$; $T_2 = 8$ | $T_n = 2(4)^{n-1}$; $T_3 = 2(4)^{3-1}$; $T_3 = 32$
b)	$4;\ \ 12;\ \ 36;$ ....
c)	$1;\ -2;\ \ 4;\ -8;$ .....
d)	$128;\ \ 64;\ \ 32;$ .....

WORKED EXAMPLE 3:

In the following sequences determine the:

a)	11^{th} term of: $-3;\ \ 6;\ -12;$ .... $T_{11} = -3072$
b)	9^{th} term of: $2;\ \ 8;\ \ 32;$ ..... $T_9 = 131072$
c)	7^{th} term of: $24;\ \ 6;\ \frac{3}{2};\ \frac{3}{8};$ ... ... $r = \frac{6}{24} = \frac{1}{4}$ $T_7 = \frac{3}{512}$

WORKED EXAMPLE 4:

a)	Which term of the geometric sequence $2;\ \ 6;\ \ 18;$ ... is equal to 4374?

	$4374 = ar^{n-1}$ $4374 = 2(3)^{n-1}$ $\frac{4374}{2} = 3^{n-1}$ $2187 = 3^{n-1}$ $3^7 = 3^{n-1}$ $7 = n - 1$ $\therefore n = 8$
b)	Find the number of terms in the sequence $2;\ \ 4;\ \ 8; \ldots .1024$ $n = 10$

WORKED EXAMPLE 5:

Determine the geometric sequence whose 4^{th} term is 24 and 7^{th} term is 192.

$T_4 = 24$ $n = 4$ $T_n = ar^{n-1}$ $24 = ar^{4-1}$ $24 = ar^3$	$T_7 = 192$ $n = 7$ $T_n = ar^{n-1}$ $192 = ar^{7-1}$ $192 = ar^6$

$\frac{24}{192} = \frac{ar^3}{ar^6}$

$\frac{1}{8} = r^{-3}$

$2^{-3} = r^{-3}$

$\therefore r = 2$

WORKED EXAMPLE 6:

The first 3 terms of the geometric sequence are given as:

	$x-4;\ \ x;\ \ x+12$
a)	Determine the value of x. $\frac{T_2}{T_1}=\frac{T_3}{T_2}$ $\frac{x}{x-4}=\frac{x+12}{x}$ $x(x)=(x+12)(x-4)$ $x^2=x^2-4x+12x-48$ $x=6$
b)	Determine the value of the first 3 terms. $T_1=x-4=6-4=2$ $T_2=\ x=6$ $T_3=\ x+12=6+12=18$
c)	Determine the value of the 10^{th} term. $T_{10}=2(3)^9$ $T_{10}=39366$

THE SUM OF A SERIES

This time we are going to determine the sum of a geometric series:

Lets see if we can find an east way to do the following series:

$$1+2+4+8+16$$

$\boldsymbol{a=1}$	$\boldsymbol{r=2}$	$\boldsymbol{n=5}$

This is an easy example. The sum is 31. But what if there are 10 or more terms in a series. Lets take a closer few to observe a pattern.

$S_5=1+2+4+8+16$ MULTIPLY SERIES WITH $r=2$

$$2S_5=\quad 2+4+8+16+32$$

$$(1)-(2): S_5 - 2S_5 = 1 - 32$$
$$-S_5 = -31$$
$$S_5 = 31$$

Now we will be doing the general term (you need to know this for examination purposes)

$$S_n = a + ar + ar^2 + ar^3 + \ldots ar^{n-1}$$
$$rS_n - S_n = ar + ar^2 + ar^3 + \ldots ar^n$$
$$S_n(r-1) = a(r^n - 1)$$
$$S_n = \frac{a(r^n-1)}{r-1}$$

WORKED EXAMPLE 7:

In the geometric series:

$4 + 12 + 36 + \cdots + 78732$

a)	Determine the number of terms. $T_n = ar^{n-1}$ $78732 = 4(3)^{n-1}$ $\frac{78732}{4} = 3^{n-1}$ $19683 = 3^{n-1}$ $3^9 = 3^{n-1}$ $\therefore n = 10$
b)	What is the sum of this series? $S_n = \frac{a(r^n-1)}{r-1}$ $S_{10} = \frac{4(3^{10}-1)}{3-1}$ $= 118096$

GRADE 12 – WORKSHEET 2

HOMEWORK ACTIVITY

1	Determine the 5^{th} and 8^{th} terms of the geometric sequence $3; 6; 12; \dots \dots$
2	Consider the geometric sequence: $x;\ \ x+3;\ \ 3x-1; \dots \dots$ Calculate the value(s) of the common ratio, r, if $r > 0$.
3	$x-4;\ \ x+2;\ \ 3x+1;$ …. Are the first three terms of a geometric sequence. Determine the sequence if x is positive.
4	If $k+1, k-1, and\ 2k-5$ are the first three terms of a geometric sequence.
a)	Determine the value of k.
b)	Determine the first four terms of the sequence.
c)	Determine the value(s) of the common ratio.
5	Determine the numerical value of the fourth term of the geometric sequence: $x; 2x+2; 3x+3$
6	$t+1;\ \ 1-t;\ \ 1-5t$ …are the first three terms of a geometric sequence
a)	Determine the numeric value of t where $t \neq 0$.
b)	Determine the sequence
c)	Determine the 10^{th} term
d)	Which term equals $10\frac{2}{3}$
7	Determine S_{23} in the geometric series $4+8+16+$…….
8	How many terms of the geometric sequence $888+444+$

	$222 + \ldots\ldots$ Will add up to $\frac{113\,553}{64}$
9	If $-\frac{1}{2}\ ;\ \frac{1}{3}\ ;k$ is a geometric sequence, determine:
a)	The common ratio
b)	The value of k.
c)	The sum of the first 6 terms.
10	The second term of a geometric sequence is 6 and the fourth term is 54. Find the sum of the series to 20 terms.