GEOMETRIC SEQUENCES AND SERIES

OUTCOMES:

1. Sum to infinity
2. Application of sequence and series

INTRODUCTION:

We have been looking at geometric series with a definite number of terms, now we are going to look at the sum of a geometric series with and infinite (∞) number of terms.

Remember the sum of a geometric series:

$$S_n = \frac{a(1-r^n)}{1-r} \quad or \quad S_n = \frac{a(r^n-1)}{r-1}$$

Compare the sum of the following geometric series:

$\mathbf{1+2+4+8+\ldots}$ $S_2 = 1+2 = 3$ $S_3 = 1+2+4 = 7$ $S_4 = 1+2+4+8 = 15$ $S_5 = 1+2+4+8+16 = 31$ $S_9 = \frac{a(r^n-1)}{r-1} = \frac{1(2^9-1)}{2-1} = 511$ $S_{20} = \frac{1(2^{20}-1)}{2-1} = 1\,048\,575$	As n **INCREASES**, S_n becomes very **LARGE**. Mathematically we say that: If $n \to \infty$ then $S_n \to \infty$ $\therefore$ This series **DIVERGES**
A geometric series will diverge (the sum will approach a very **LARGE** positive or negative value) as n approaches infinity.	

$$\mathbf{1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots}$$

$S_2 = 1 + \frac{1}{2} = 1{,}5$

$S_3 = 1 + \frac{1}{2} + \frac{1}{4} = 1{,}75$

$S_4 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 1{,}875$

$S_5 = 1{,}9375$

$S_6 = 1{,}984375$

$S_9 = 1{,}996\ldots\ldots \approx 2$

As n **INCREASES**, S_n approaches 2.

Mathematically we say that:

If $n \to \infty$ then $S_n \to 2$

$\therefore$ This series **CONVERGES** to2

A geometric series will converge (the sum will approach a specific value) as n approaches infinity, only if the constant ratio is a number between -1 and 1.

Thus:

Convergent geometric series

$$\boldsymbol{-1 < r < 1 \text{ and } r \neq 0}$$

A geometric series will only converge if $-1 < r < 1$. We can derive a formula for the sum to infinity as follows:

$$\boldsymbol{S_\infty = \frac{a}{1-r}}$$

WORKED EXAMPLE 1:

Determine the sum to infinity for the following series:

a)	$27 + 9 + 3 + 1 + \ldots\ldots\ldots\ldots$ $a = 27$ $r = \frac{9}{27} = \frac{1}{3}$ $S_\infty = \frac{a}{1-r}$ $S_\infty = \frac{27}{1-\left(\frac{1}{3}\right)}$ $S_\infty = \frac{81}{2}$
b)	$16 - 4 + 1 - \frac{1}{4} + \ldots\ldots\ldots\ldots\ldots\ldots$ $a = 16$ $r = -\frac{1}{4}$ $S_\infty = \frac{a}{1-r}$ $S_\infty = \frac{16}{1-\left(-\frac{1}{4}\right)}$ $S_\infty = 12.8$
c)	$\sum_{n=1}^{\infty} 2.\left(\frac{1}{2}\right)^{n-1}$ $T_1 = 2\left(\frac{1}{2}\right)^{1-1} = 2$ $T_2 = 2\left(\frac{1}{2}\right)^{2-1} = 1$

	$T_3 = 2\left(\frac{1}{2}\right)^{3-1} = \frac{1}{2}$ $a = 2$ $r = \frac{1}{2}$ $S_\infty = \frac{a}{1-r}$ $S_\infty = \frac{2}{1-\frac{1}{2}}$ $S_\infty = 4$
WORKED EXAMPLE 2: Given the geometric series: $256 + p + 64 - 32 + \cdots$	
a)	Determine the value of p. $r = \frac{T_2}{T_1} = \frac{T_3}{T_2} = \frac{T_4}{T_3}$ $\frac{p}{256} = \frac{-32}{64}$ $p = -128$

b)	Calculate the sum of the first 8 terms of the series. $r = \frac{-128}{156} = -\frac{1}{2}$ $S_n = \frac{a(1-r^n)}{1-r}$ $S_n = \frac{256\left(1-\left(-\frac{1}{2}\right)^8\right)}{1-\left(-\frac{1}{2}\right)}$ $S_n = 170$
c)	Why does the sum to infinity for this series exist? $-1 < r < 1$
d)	Calculate S_∞. $S_\infty = \frac{a}{1-r}$ $S_\infty = \frac{256}{1-\left(-\frac{1}{2}\right)}$ $S_\infty = \frac{512}{3}$

WORKED EXAMPLE 3:

For which values of x will the series converge?

$(x+1) + (x+1)^2 + (x+1)^3 + \cdots$

Convergent series:

$$r = \frac{T_2}{T_1} = \frac{(x+1)^2}{(x+1)} = x+1$$

For this series to converge; $-1 < r < 1$

$\therefore -1 < x+1 < 1$

$\therefore -1 - 1 < x + 1 < 1 - 1$ $\therefore -2 < x < 0$	
WORKED EXAMPLE 4: APPLICATION Consider the following sequence of numbers: $5;\ \ 4;\ \ 5;\ \ 11;\ \ 5;\ \ 18;\ \ 5;\ \ 25;\ldots$	
a)	Given that the pattern continues. Write down the next two terms of the sequence. THE PATTERN CONSIST OF SEQUENCES $5;\ \ 4;\ \ 5;\ \ 11;\ \ 5;\ \ 18;\ \ 5;\ \ 25;\ldots$ **UNEVEN TERMS** ARE CONSTANT: $T_1 = T_3 = T_5 = 5$ **EVEN TERMS** FORMS AN ARITHMETIC SEQUENCE; WHERE: $4;\ \ 11;\ \ 18;\ \ 25$ $a = 4;\quad d = 7$ The next terms are 5 and 32
b)	Determine the sum of the first 50 terms of the sequence. $S_{50} = S_{(25)uneven} + S_{(25)even}$ $S_{50} = 25 \times 5 + \frac{25}{2}[2(4) + (25 - 1)(7)]$ $S_{50} = 125 + 2\ 200$ $S_{50} = 2\ 325$
WORKED EXAMPLE 5: APPLICATION Thabo trains to run the Two ocean marathon. He runs 4 km in the first week and increase his distance by 3 km each week after that. **METHOD:** 1) Read the question and find the sequence first. 2) Determine what type of sequence Arithmetic or Geometric.	

3) Now apply the formula depending which sequence it is.	
a)	What distance did he run during the 11th week? Sequence: 4; 7; 10 .... $T_n = a + d(n-1)$ $T_n = 4 + 3(n-1)$ $T_n = 4 + 3n - 3$ $T_n = 3n + 1$ $T_{11} = 3(11) + 1 = 34$ ∴He ran 34 km in 11 weeks
b)	During which week did he run 25 km? $T_n = 25$ $T_n = 3n + 1$ $25 = 3n + 1$ $24 = 3n$ $\therefore n = 8$ ∴He ran 25 km in week 8.
c)	What is the total distance he runs on the first 15 weeks. $n = 15$ $S_n = \frac{n}{2}[2a + d(n-1)]$ $S_{15} = \frac{15}{2}[2(4) + (3)(15-1)]$ $S_{15} = 375$ ∴ He ran 375 km in the first 15 weeks
d)	It is said that an athlete run at least 900 km in preparation for the Two oceans marathon. How many weeks should Thabo train

before he meets his goal?

$S_n = 900$

$S_n = \frac{n}{2}[2a + d(n-1)]$

$900 = \frac{n}{2}[2(4) + 3(n-1)]$

$1800 = n[8 + 3n - 3]$

$1800 = 3n^2 + 5n$

$\therefore 0 = 3n^2 + 5n - 1800$

$\therefore n = \frac{-5 \pm \sqrt{(5)^2 - 4(3)(-1800)}}{2(3)}$

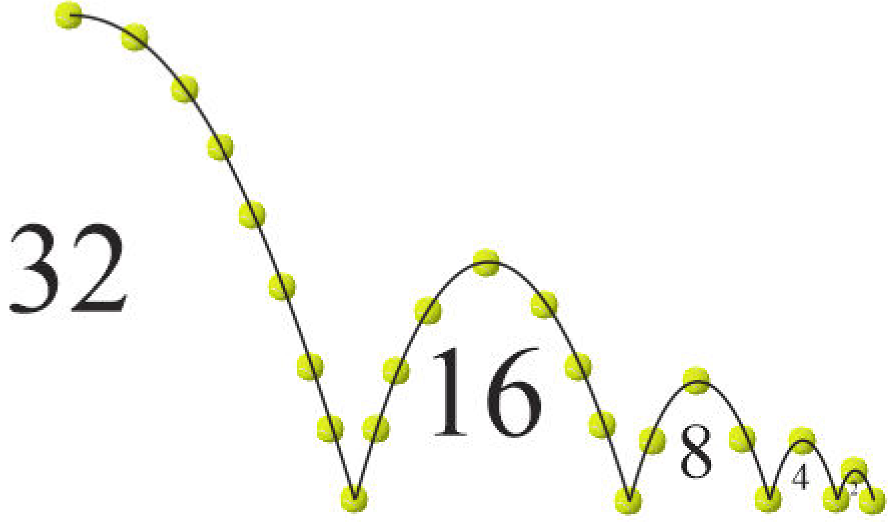
$\therefore n = 23{,}68$ or $n = -25{,}34$

$\therefore$He will have to train for 24 weeks.

WORKED EXAMPLE 6: APPLICATION

A ball is dropped from a height of 32 m and bounces continuously. With each successive bounce, the ball reaches a height that is 50% of the previous height. If this motion continues indefinitely, what is the total vertical distance travelled by the ball over its entire journey?

Solution:



From the sketch we can write the sequence as:

Vertical Distance
$= 32 + 16 + 16 + 8 + 8 + 4 + 2 + \ldots$ **(infinite geometric series)**
$= 32 + 2(16) + 2(8) + 2(4) + ..$
$= 32 + 2(16 + 8 + 4 + ..)$
$= 32 + 2\left(\frac{16}{1 - 0,5}\right)$
$= 32 + 2(32)$
$= 96\text{ m}$

HOMEWORK ACTIVITY

1	Determine the sum to infinity for the following series:
a)	$-32 + 16 - 8 + 4 + \ldots\ldots\ldots\ldots$
b)	$16 + 8 + 4 + \cdots$
c)	$48 - 12 + 3 - \cdots$
d)	$\sum_{n=1}^{\infty} 2.(4)^{1-n}$
e)	$\sum_{n=1}^{\infty} 2.\left(\frac{1}{3}\right)^{n-1}$
f)	$\sum_{n=1}^{\infty} \left(-\frac{4}{5}\right)^{n-1}$
2	Given the sequence $5(4^5) + 5(4^4) + 5(4^3) + \ldots\ldots\ldots\ldots$
a)	Show that the series is convergent.
b)	Calculate the sum to infinity of the series.
3	Given the geometric series $9x + 3x^2 + x^2 \ldots\ldots\ldots$

a)	Show that $T_n = 27\left(\frac{x}{3}\right)^n$
b)	For which values of x will the series converge?
c)	Calculate the sum to infinity if $x = 2$.
4	Determine the value of x for which the following series converges
a)	$x + 2x^2 + 4x^3 + \ldots$
b)	$(2x - 5) + (2x - 5)^2 + (2x - 5)^3 + ..$
5	Consider the infinite series: $45 + 40{,}5 + 36{,}45 + ..$
a)	Calculate the value of the TWELTH term of the series (correct to two decimal places).
b)	Explain why the series converges.
c)	Calculate the sum to infinity of the series.
6	Thabo trains for the Spar cycle race. He cycles a distance of 50 km in the first week and increases his distance by 15 km each week after that.
a)	What distance did he cycle during the 15^{th} week? Answer: $260\ km$
b)	During which week did he run 185 km? Answer: $n = 10$
c)	What is the total distance he cycled over a 20 week period? Answer: $3850\ km$
d)	Thabo first started cycling 15 weeks before the end of the year. His goal is to cycle 3000 km before the end of the year. Will he be

	able to reach his goal? Answer: $S_{15} = 2325\ km$ ∴No, he will not reach his goal
7	A tree with an initial height of 2m is planted. In the first year it grows 1,5m, in the second year it grows 1,2m, in the third year 0,96m, in the fourth year 0,768m and so forth. What is the maximum height that the tree can reach? Answer:$9,5m$