EUCLIDEAN GEOMETRY: SIMILARITY

OUTCOMES:

1. Equiangular triangles (all three angles equal) are similar
2. If the corresponding sides of two triangles are in the same proportion, then the triangles are similar

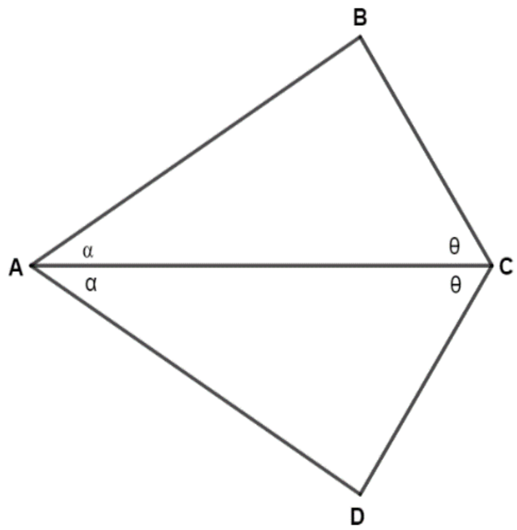
INTRODUCTION:

Let's go back to the previous grades and review congruency for triangles.

CONGRUENCY($\equiv$): Two triangles are congruent if they have the same shape and size. Thus the two triangles are identical, the corresponding angles are the same and the corresponding sides are equal.

Two triangles are congruent if they have:

- 3 sides the same length: (S; S; S)
- 2 sides and an included angle: (S; A; S)
- 2 angles and a side equal: (A; A; S)
- A right angle, hypotenuse and a side equal: (R; H; S) INVB



In ΔABC and $\Delta\ ADC$:

1. $B\hat{A}C = D\hat{A}C$ given
2. $B\hat{C}A = D\hat{C}A$ given
3. AC is common

$\therefore\ \Delta ABC \equiv \Delta\ ADC$ (A; A; S)

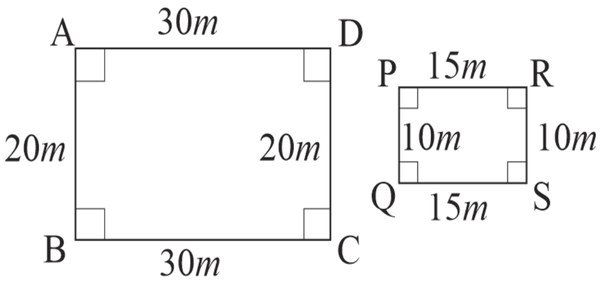
Since they are congruent, we can say now that: $\hat{B} = \hat{D}$; $AB = AD$ and $BC = DC$

SIMILARITY(|||): Polygons (A **polygon** is a flat, two-dimensional (2D) **shape** with straight sides that is fully closed, all the sides are joined up) are **similar** if they have the **same shape**. If two polygons are similar, the one is an **enlargement** of the other.

Two polygons are similar if and only if:

- All pairs of corresponding angles are equal **AND**
- All pairs of corresponding sides are in the same proportion. **Both** of these conditions have to be met for two polygons to be similar.

Example: These two rectangles are similar to each other as are the three pentagons.



Note $\frac{AD}{PR}=\frac{AB}{PQ}=\frac{BC}{QS}=\frac{CD}{RS}=2$ $\therefore$ corresponding sides are in proportion.

$\hat{A}=\hat{P}$, $\hat{B}=\hat{Q}$, $\hat{C}=\hat{S}$ and $\hat{D}=\hat{R}$ $\therefore$ equiangular as corresponding angles are equal.

$\therefore$ the two rectangles are similar to each other.

We write this a ABCD ||| PQRS.

For Triangles, any one of these two conditions is sufficient to guarantee similarity.

Hence in any two triangles if:

1. All pairs of corresponding angles are equal, then the two triangles are similar, **OR**
2. If all pairs of corresponding sides are in the same proportion, then the two triangles are similar.

If one of the conditions is true for two triangles, then the other condition is automatically also true.

NOTATION

$\Delta XYZ \mid\mid\mid \Delta MNP$ means 'triangle XYZ is similar to triangle MNP'. The **order** in which the letters are written is very important, as it indicates **which angles are equal.**

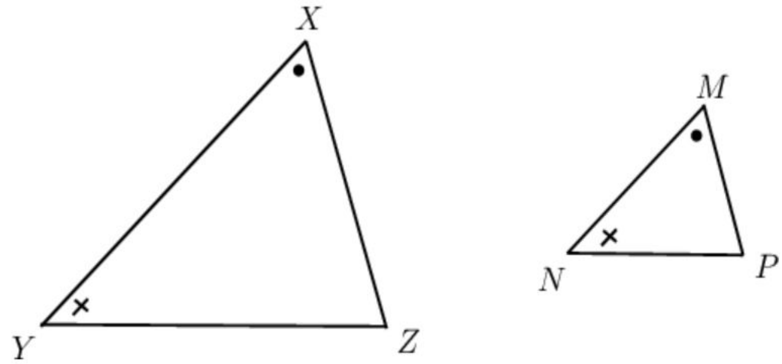
Hence:

$\hat{X} = \hat{M}$, $\hat{Y} = \hat{N}$ and $\hat{Z} = \hat{P}$.

The order also indicates which **ratios of sides are equal**: ΔXYZ ||| Δ MNP

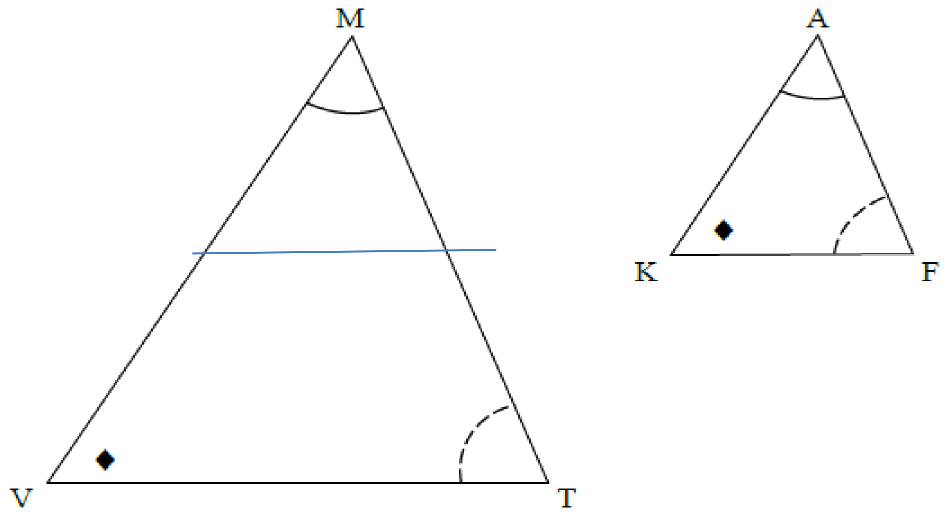
$\frac{XY}{MN} = \frac{YZ}{NP} = \frac{XZ}{MP}$ or $\frac{XY}{YZ} = \frac{MN}{NP}$

or $\frac{XY}{XZ} = \frac{MN}{MP}$ or $\frac{YZ}{XZ} = \frac{NP}{MP}$



WORKED EXAMPLE 1:

In the diagram below, ΔMVT and ΔAKF are drawn such that $\hat{M} = \hat{A}$, $\hat{V} = \hat{K}$ and $\hat{T} = \hat{F}$



Use the diagram to prove the theorem which states that if two triangles are equiangular, then the corresponding sides are in proportion, that is $\frac{MV}{AK} = \frac{MT}{AF}$

Construction:

Draw line BC such the MB = AK and MC = AF

Proof:

In ΔBMC and ΔKAF

MB = AK [Construction]

$\hat{M} = \hat{A}$ [Given]

MC = AF [Construction]

ΔBMC ≡ ΔKAF [s∠s]

$\therefore M\hat{B}C = A\hat{K}F$ or $M\hat{C}B = A\hat{F}K$ [≡ Δ]

BUT:

$\hat{V} = \hat{K}$ or $\hat{T} = \hat{F}$ [Given]

$\therefore M\hat{B}C = \hat{V}$ or $M\hat{C}B = \hat{T}$

But these are corresponding ∠s

$\therefore BC||VT$ [corresponding $\angle s$ =]

$\therefore \frac{MV}{MB} = \frac{MT}{MC}$ [proportionality theorem; $BC||VT$]

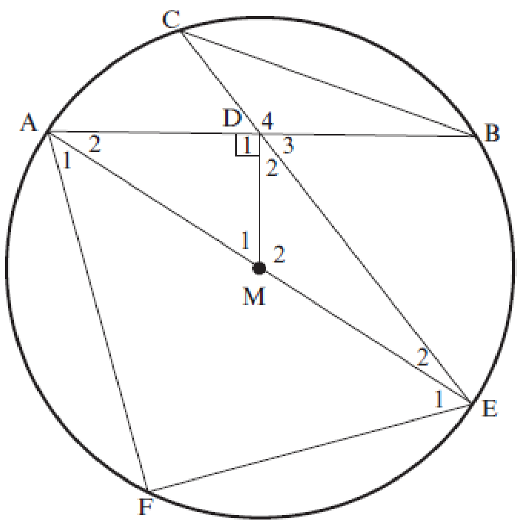
BUT:

$MB = AK$ and $MC = AF$ [construction]

$\therefore \frac{MV}{AK} = \frac{MT}{AF}$

WORKED EXAMPLE 2:

Diameter AME of circle with centre M bisects $F\hat{A}B$. MD is perpendicular to the chord AB. ED produced meets the circle at C, and CB is joined.



a)	Prove ΔAEF ||| ΔAMD (5) **HINT: Highlight the triangles you working with**

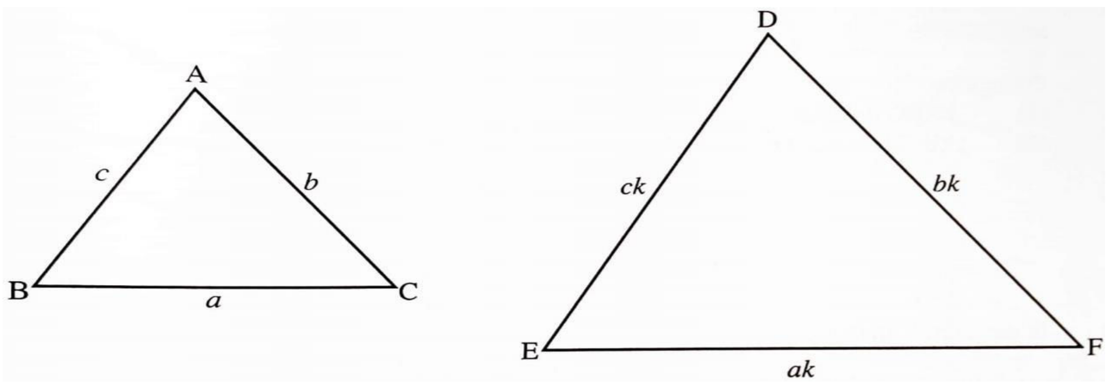
	Statement	**Reason**
	$\hat{F} = 90°$	(∠ in semi-circle)
	$\hat{D}_1 = 90°$	(given MD ⊥ AB)
	$\therefore\ \hat{F} = \hat{D}_1$	(both equal to 90°)
	In ΔAEF and ΔAMD	
	$\hat{F} = \hat{D}_1$	(proved)
	$\hat{A}_1 = \hat{A}_2$	(AM bisects $F\hat{A}B$)
	$\therefore\ \hat{E}_1 = \hat{M}_1$	(third ∠ of Δ)
	∴ ΔAEF ||| ΔAMD	(∠∠∠)
b)	Hence, find the numerical value of $\frac{AF}{AD}$	(5)
	Statement	**Reason**
	$\frac{AE}{AM} = \frac{EF}{MD} = \frac{AF}{AD}$	(||| Δs)
	$AM = ME$	(radii)
	$\therefore\ AE = 2AM$	
	$\therefore\ \frac{2AM}{AM} = \frac{AF}{AD}$	
	$\frac{AF}{AD} = 2$	
c)	Prove Δ CDB ||| ΔADE	(4)
	Statement	**Reason**
	In ΔCDB and ΔADE	
	HINT: Highlight the triangles you working with	
	$\hat{C} = \hat{A}_2$	(∠s in same segment, arc BE)
	$\hat{B} = \hat{E}_2$	(∠s in same segment, arc AC)
	$\hat{D}_4 = \hat{D}_1 + \hat{D}_2$	(vert opp. ∠ s =)
	∴ ΔCDB ||| ΔADE (∠∠∠)	

d) Prove $AD^2 = CD.DE$ (3)

Statement	**Reason**
$\frac{CD}{AD} = \frac{DB}{DE} = \frac{CB}{AE}$	(||| Δs)
$\therefore CD.DE = AD.DB$	
But $AD = DB$	(MD ⊥ AB, M is centre, thm 1
$\therefore CD.DE = AD.AD$	
$\therefore AD^2 = CD.DE$	

Using sides to prove that triangles are similar

If the corresponding sides of two triangles are in the same proportion, then the triangles are equiangular and hence similar.



Given: **ΔABC** and **ΔDEF** with

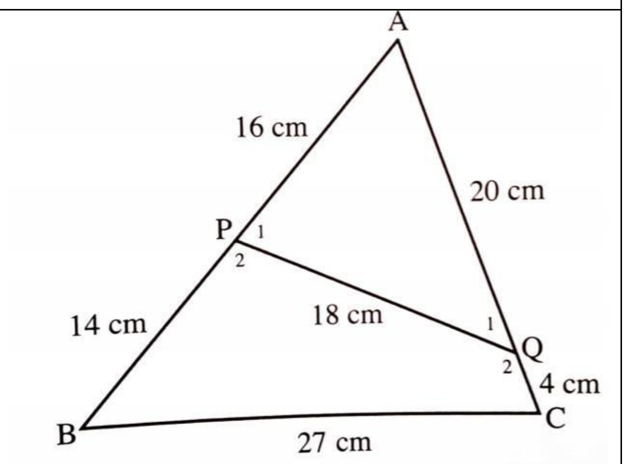
$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$$

Conclusion: $\hat{A} = \hat{D}$, $\hat{B} = \hat{E}$, $\hat{C} = \hat{F}$ and hence **ΔABC ||| Δ DEF** .
Reason: sides of Δs in prop

WORKED EXAMPLE 3:

In the sketch below, $AP = 16cm$, $PB = 14cm$ $AQ = 20cm$, $QC = 4cm$ and $BC = 27cm$.

Prove that :



a) $\Delta APQ \ ||| \ \Delta ACB$

$$\frac{AP}{AC} = \frac{16cm}{24cm} = \frac{2}{3}$$

$$\frac{PQ}{CB} = \frac{18cm}{27cm} = \frac{2}{3}$$

$$\frac{AQ}{AB} = \frac{20cm}{30cm} = \frac{2}{3}$$

$$\therefore \frac{AP}{AC} = \frac{PQ}{CB} = \frac{AQ}{AB}$$

$\therefore \Delta APQ \ ||| \ \Delta ACB$ (sides of Δs in prop.)

b) PBCQ is a cyclic quadrilateral.

Highlight the cyclic quadrilateral you working with

$\hat{P}_1 = \hat{C}$ (||| Δs) [AP_1Q_1 ||| ACB, MEANING:

$\angle A = \angle A$

$\angle P_1 = \angle C$

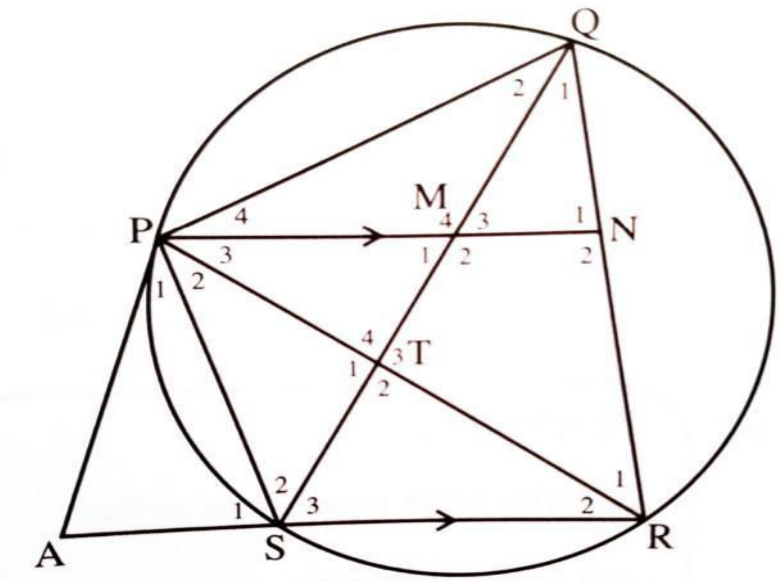
$\angle Q_1 = \angle B$

∴ PBCQ is a cyclic quad. (ext ∠ of quad = int opp∠)

WORKED EXAMPLE 4:

In the sketch alongside, AP is a tangent to the circle at P. PN ∥ SR

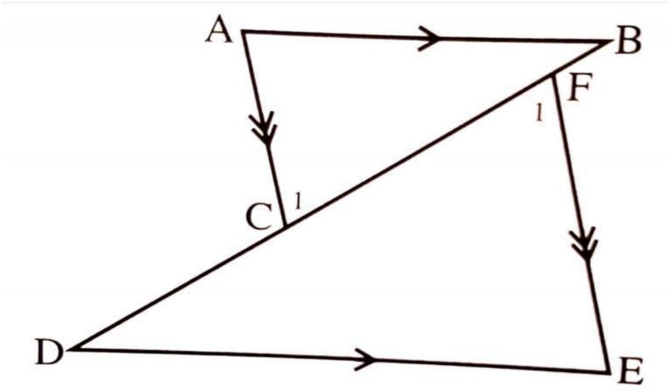
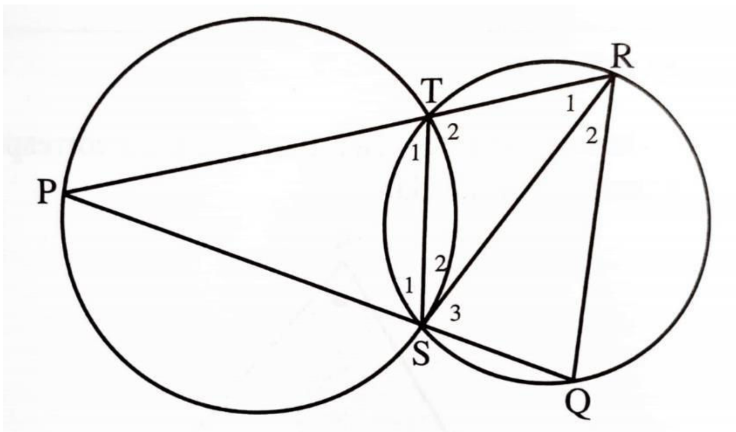
Prove that :

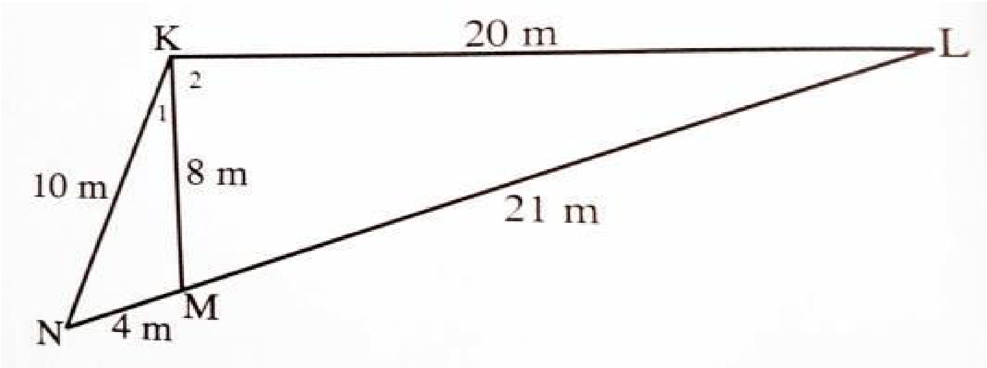
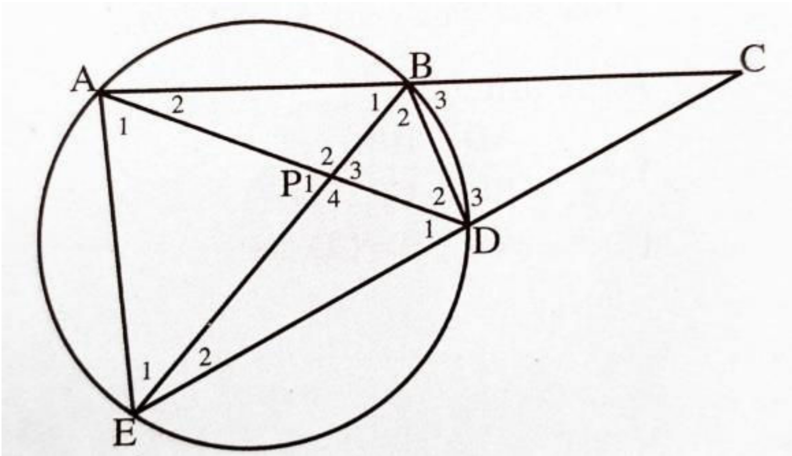


a)	$\frac{PS}{QR} = \frac{ST}{RT}$ **ΔPST and ΔQRT are both triangles in the sketch, hence we will attempt to prove that ΔPST and ΔQRT are similar.** **Statement** — **Reason** In ΔPST and ΔQRT: $\hat{P}_2 = \hat{Q}_1$ — (∠s in same segment also you can say subtended chord SR $\hat{S}_2 = \hat{R}_1$ — (∠s in same segment also you can say subtended chord PQ) $\hat{T}_1 = \hat{T}_3$ — (3^{rd}∠ of Δ) ∴ ΔPST ||| Δ QRT (∠∠∠) **PAY SPECIAL ATTENTION TO THE ORDER OF THE LETTERS** $\therefore \frac{PS}{QR} = \frac{ST}{RT}$ — (||| Δs)
b)	$\frac{PQ}{PT} = \frac{SR}{ST}$ **ΔPQT and ΔRST are both triangles in the sketch, hence we will attempt to prove them similar.** **Statement** — **Reason** In ΔPQT and ΔRST: $\hat{P}_3 + \hat{P}_4 = \hat{S}$ — (∠s in same segment) $\hat{Q}_2 = \hat{R}_2$ — (∠s in same segment) $\hat{T}_4 = \hat{T}_2$ — (3^{rd}∠ of Δ) ∴ ΔPQT ||| Δ SRT (∠∠∠) **PAY SPECIAL ATTENTION TO THE ORDER OF THE LETTERS** $\therefore \frac{PQ}{PT} = \frac{SR}{ST}$ — (||| Δs)
c)	PM . ST = RS . MT

	Prove that ΔMPT and ΔRST are similar. **Statement** — **Reason** In ΔMPT and ΔRST: $\hat{P}_3 = \hat{R}_2$ — (alt ∠s ; PN ∥ SR) $\hat{M}_1 = \hat{S}_3$ — (alt ∠s ; PN ∥ SR) $\hat{T}_4 = \hat{T}_2$ — (3^{rd}∠ of Δ) $\therefore \Delta PMT \;\vert\vert\vert\; \Delta RST \quad (\angle\angle\angle)$ $\therefore \frac{PM}{RS} = \frac{MT}{ST}$ $\therefore PM \,.\, ST = RS.\, MT$
d)	$AP^2 = AR \,.\, AS$ **Prove that ΔAPS and ΔAPR are similar.** **Statement** — **Reason** In ΔAPS and ΔAPR: $\hat{P}_1 = \hat{R}_2$ — (tan chord thm) $\hat{A} = \hat{A}$ — (common) $\hat{S}_1 = \hat{P}_1 + \hat{P}_2$ — (3^{rd}∠ of Δ) $\therefore \Delta PAS \;\vert\vert\vert\; \Delta RAP$ — (∠∠∠) $\therefore \frac{PA}{RA} = \frac{AS}{AP}$ — (||| Δs) $\therefore AP^2 = AR.\, AS$

HOMEWORK ACTIVITY

1	In the sketch below, $AB \parallel DE$ and $AC \parallel FE$  Prove that:
a)	ΔBCA ||| ΔDFE
b)	$AB.EF = AC.ED$
2	In the sketch below, SR is a tangent to circle PST at S.  Prove that:
a)	ΔPRS ||| ΔSRT
b)	$RS^2 = PR.RT$
c)	ΔPQR ||| ΔPTS

d)	$\frac{\text{RT}}{\text{PT}} = \frac{\text{RS}^2}{PQ\ .\ PS}$
3	In the sketch alongside, KL= 20m, KN = 10m MN = 4m, KM = 8m and LM= 21m.  Prove that :
a)	ΔKMN ||| Δ LKN
b)	KN is a tangent to circle LMK at K.
4	In the sketch below, prove that 
a)	$\frac{\text{AB}}{\text{ED}} = \frac{\text{AP}}{\text{EP}}$
b)	AE . CD = AC . BD

PROOF