GRADE 12 – WORKSHEET 9

TRIGONOMETRY 3 – GENERAL SOLUTION

OUTCOMES:

1. General solution

INTRODUCTION - OVERVIEW OF BASIC CONCEPTS

The trigonometric functions $\sin\theta$ and $\cos\theta$ have a **period of 360°** and $\tan\theta$ has a **period of 180°**. The **period** is the number of degrees needed for a trigonometric function to complete one cycle.

This means for $\sin\theta$ and $\cos\theta$ the same numerical value will be obtained when adding or subtracting $360°$ to the specific angle. For $\tan\theta$, the same numerical value will be obtained when $180°$ is added or subtracted to the specific angle.

The concept of a general solution:

Look at the solution of $\sin\theta = \frac{1}{2}$

- $\sin 30° = \frac{1}{2}$
- $\sin 390° = \frac{1}{2} = [\sin(30° + \mathbf{1} \times 360°)]$
- $\sin 750° = \frac{1}{2} = [\sin(30° + 360° + 360°)] = \sin(30° + \mathbf{2} \times 360°)$
- $\sin 1110° = \frac{1}{2} = [\sin(30° + 360° + 360° + 360°)] = \sin(30° + \mathbf{3} \times 360°)$

YOU CAN SUMMARISE THE GENERAL SOLUTION TO A TRIGONOMETRIC EQUATIONS OF:

$\sin x = \frac{1}{2}$:

$x = 30° + n.360\, , n \in \mathbb{Z}$ or $x = 150 + n.360\, , n \in \mathbb{Z}$

THE SET OF INTEGERS, $\mathbb{Z}$

This infinite set contains negative number and positive whole numbers.

- $\mathbb{Z} = \{..; -5;\ -4;\ -3;\ -2;\ -1; 0; 1; 2; 3; 4..\}$

WORKED EXAMPLE 1:

a

Determine the general solution of $\sin\theta = +\frac{1}{2}$

$\sin\theta$ is positive in Quadrant 1 and 2

Reference angle$= \sin^{-1}\left(\frac{1}{2}\right) = 30°$

QUADRANT 1

$\theta =$ Reference angle $+k.360°$

$\therefore \theta = 30° + k.360°$ with $k \in \mathbb{Z}$

QUADRANT 2

$\theta = 180° - RA + k.360°$

$\therefore \theta = 180° - 30° + k.360$

$\therefore \theta = 150° + k.360°$ with $k \in \mathbb{Z}$

The adding of $k.360°$ means we are adding multiples of $360°$ in any direction.

b

Find θ if $\theta\epsilon(-360°; 360°)$

	$k = -1$	$k = 0$	$k = 1$
$\boldsymbol{\theta = 30° + k.360°}$	$-330°$	$30°$	$\mathbf{390°}$
$\boldsymbol{\theta = 150° + k.360°}$	$-210°$	$150°$	$\mathbf{510°}$

$\therefore \theta = -330°$ or $-210°$ or $30°$ or $150°$

The other angles don't fall inside the interval $\boldsymbol{\theta\epsilon(-360°; 360°)}$

WORKED EXAMPLE 2:

Solve for x, given $\sin x = 3\cos x$ where $\boldsymbol{x} \in (\mathbf{-360°; 360°})$

$\sin x = 3\cos x$

$\frac{\sin x}{\cos x} = \frac{3\cos x}{\cos x}$ **[Divide by $\cos x$ to isolate the ratio]**

$\tan x = 3$ **[Tan is positive - CAST rule]**

Reference Angle: $x = 71{,}57°$

Quadrant 1 $x = RA = 71{,}57° + k.180° for\ k \in Z$
Quadrant 3 $x = 180° + 71.57° + k.180\ \ for\ k \in Z$ $\therefore x = 251.57° + k.180°\ for\ k \in Z$

Specific Solution: Choose different values for k to determine solutions within the interval:

For $k = -2; -1; 0; 1; 2$

$Then: -288.43°;\ \ -108.43°;\ \ 71.47°; 251.57°$

Only needed if restrictions are given!

WORKED EXAMPLE 3:

Determine the general solution for:

$\sin 2x . \cos x + \cos 2x . \sin x = \cos 60°$ **[Compound rule on LHS]**

Solution:

$\boldsymbol{\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta}$

$\sin(2x + x) = \frac{1}{2}$

$\sin 3x = \frac{1}{2}$

Reference Angle = 30°

Quadrant 1

$2x + x = 30° + k.360°\ for\ k \in Z$

$3x = 30° + k.360°\ for\ k \in Z$

$x = 10° + k.120° for\ k \in Z$

Quadrant 2

$2x + x = 150° + k.360° for\ k \in Z$

$3x = 150° + k.360° for\ k \in Z$

$x = 50° + k.120° for\ k \in Z$

WORKED EXAMPLE 4:

Determine the general solution of $2\sin 2x + 3\sin x = 0$

Solution:

$2\sin 2x + 3\sin x = 0$ **[Double Angle - expand]**

$2(2\sin x\,.\cos x) + 3\sin x = 0$ [**Remove brackets**]

$4\sin x\,.\cos x + 3\sin x = 0$ [**factorise – common factor**]

$\sin x\,(4\cos x + 3) = 0$

$\sin x = +0$

$RA = \sin^{-1} 0°$

$Ref\ Angle = 0°$

Quadrant 1[sin positive]

$\boldsymbol{x = 0° + k.360°\,; k \in Z}$

Quadrant 2 [sin positive]

$\boldsymbol{x = 180° - 0° + k.360°; k \in Z}$

$\boldsymbol{x = 180° + k.360°; k \in Z}$

$\cos x = -\frac{3}{4}$ [**Be careful of the negative sign**]

$RA = \cos^{-1}\left(\frac{3}{4}\right)$

$Ref\ Angle = 41.41°$

Quad 2: [cos negative]

$x = 180° - 41{,}41° = 138{,}59°$

$\boldsymbol{x = 138,59° + k.360°\ ;\ k \in Z}$

Quad 3: [cos negative]

$x = 180° + 41{,}41° + k.360°; k \in Z$

$\boldsymbol{x = 221,41° + k.360°;\ \ k \in Z}$

WORKED EXAMPLE 5:

Determine the general solution of $\cos 2x + \cos x = 0$

Solution:

$\boldsymbol{\cos 2A = \cos^2 A - \sin^2 A}$

$\boldsymbol{\cos 2A = 2\cos^2 A - 1}$

$\boldsymbol{\cos 2A = 1 - 2\sin^2 A}$

$\cos 2x + \cos x = 0$ **[Double Angle- expand]**

$2\cos^2 x - 1 + \cos x = 0$ **[trinomial**)

$2\cos^2 x + \cos x - 1 = 0$ [**standard form**]

$(2\cos x - 1)(\cos x + 1) = 0$ **[Factorise]**

$\cos x = \frac{1}{2}$

$Ref\ Angle = 60°$

Quad 1 [cos positive]

$x = 60° + k.360°\ ; k \in Z$

Quad 4 [cos positive]

$x = 360° - 60° + k.360°\ ; k \in Z$

$\boldsymbol{x = 300 + k.360°\ ; k \in Z}$

$\cos x = -1$

$Ref\ Angle = 0°$

Quad 2

$x = 180° - 0 + k.360°\ ; k \in Z$

$x = 180° + k.360°\ ; k \in Z$

Quad 3

$x = 180° - 0 + k.360°\ ; k \in Z$

$x = 180° + k.360°\ ; k \in Z$

HOMEWORK ACTIVITY

1	Determine the general solution of:
a)	$\cos x = 0{,}789$
b)	$\cos 3x = -0{,}123$
c)	$\sin(x - 16°) = 0{,}616$
d)	$3\cos(x - 15°) + 1 = -0{,}456$
e)	$2\tan(2x - 10°) = 10{,}67$
f)	$5\sin^2 x - 3\sin x - 2 = 0$
g)	$10\cos^2 x - \cos x - 2 = 0$
h)	$4\sin^2 x + \sin x = 3$
i)	$\cos x = 2\sin 75°.\cos 75°$ $x = 60° + k.360; k \in Z\ or$ $x = 300° + k.360°; k \in Z$
j)	$\sin 2x = 1 - \cos 2x$ $x = 0° + k.360°; k \in Z$ $x = 45° + k.360°; k \in Z;$
k)	$\cos 2x = \sin x + 1$ $x = 0° + k.360°; k \in Z$ $x = 270° + k.360°; k \in Z;$

l)	$\cos 2x + 1 = \cos x$ $x = 90° + k.360°\ or 270° + k.360°; k \in Z$ $x = 60° + k.360°\ or\ 300° + k.360°; k \in Z$
m)	$\sin x - \sin 2x = 0$ $x = 0° + k.360°\ or\ 180° + 360°; k \in Z$ $x = 60° + k.360°\ or\ 300° + k.360°; k \in Z$
n)	$\cos^2 x = 2 \sin x$ $x = 90° + k.360°\ or 270° + k.360°; k \in Z$ $x = 9{,}46° + k.180°; k \in Z$
o)	$\cos 2x - 7 \cos x - 3 = 0$ $x = 120° + k.360°\ \ or\ x = 240° + k.360°; k\epsilon Z$
p)	$\cos 2x + 4 \sin^2 x = 5 \sin x + 3$ $x = 199° + k.360°\ \ or\ x = 340{,}53° + k.360°; k \in Z$
q)	$2 \sin^2 x + \sin x = 3$ $x = 90° + k.360°\ ; k\epsilon Z$