ANALYTICAL GEOMETRY

OUTCOMES:

- The equation of a circle with centre the origin and radius r
- The equation of a circle with centre (a;b) and radius r
- The equation of a tangent to a circle

INTRODUCTION - OVERVIEW OF BASIC CONCEPTS

The equation of a circle with centre the origin and radius r

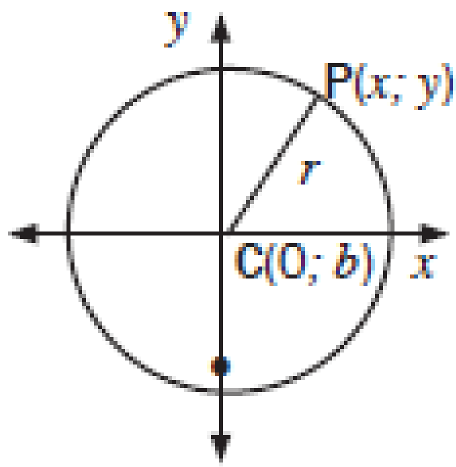
$$\boldsymbol{x^2 + y^2 = r^2}$$

We use the distance formula to work out the equation of a circle with centre $C(0;0)$. If $P(x;y)$ is any point on the circle with radius r then:

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$r = \sqrt{(x - 0)^2 + (y - 0)^2}$

$r^2 = x^2 + y^2$



WORKED EXAMPLE 1:

Find the equation of a circle centre O with the point $P(5;2)$ on its circumference.

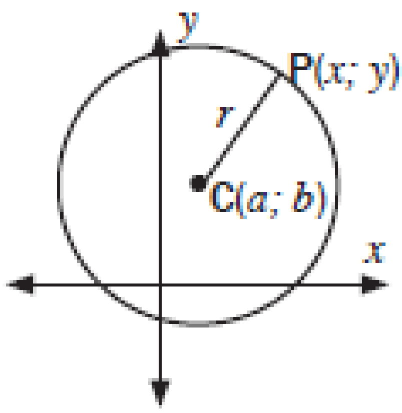
$r^2 = x^2 + y^2$
$r^2 = 5^2 + 2^2$

$r^2 = 29$

$\therefore x^2 + y^2 = 29$

The equation of a circle with centre (a;b) and radius r

$$(x-a)^2+(y-b)^2=r^2$$

If we move the centre of the circle to any point on the Cartesian plane $C(a;b)$. Then: $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ $r=\sqrt{(x-a)^2+(y-b)^2}$ $r^2=(x-a)^2+(y-b)^2$	

WORKED EXAMPLE 2:

The equation of a circle is $(x+1)^2+(y-3)^2=16$.

Determine the coordinates of the centre and the length of the radius.

The equation is already in the form $r^2=(x-a)^2+(y-b)^2$, with $a=-1, b=3$ and $r^2=16$.

So: the centre is $(-1;3)$ and the radius is:

$r^2=16 \therefore r=\pm 4 \therefore r=4$

NOTE: the radius can only be a positive number because it is a length.

WORKED EXAMPLE 3:

Determine the equation of a circle with centre $C(-1;-2)$ and passing through the point $B(1;-6)$.

First find the value of r^2

$r^2=(x-a)^2+(y-b)^2$

STEP 1: Substitute centre $C(-1;-2)$

$r^2 = (x+1)^2 + (y+2)^2$

STEP 2: Substitute $B(1;-6)$

$r^2 = (1+1)^2 + (-6+2)^2$

$r^2 = (2)^2 + (-4)^2$

$r^2 = 20$

STEP 3: Replace r^2 and centre

$\therefore 20 = (x+1)^2 + (y+2)^2$

WORKED EXAMPLE 4:

Determine the coordinates of the centre and the length of the radius if a circle has the equation: $x^2 - 2x + y^2 + 10y = -14$

$(x^2 - 2x) + (y^2 + 10y) = -14$

$(x^2 - 2x + 1) - 1 + (y^2 + 10y + 25) - 25 = -14$

$(x^2 - 2x + 1) + (y^2 + 10y + 25) = -14 + 1 + 25$

$(x-1)(x-1) + (y+5)(y+5) = 12$

$(x-1)^2 + (y+5)^2 = 12$

The centre is: $(1;-5)$

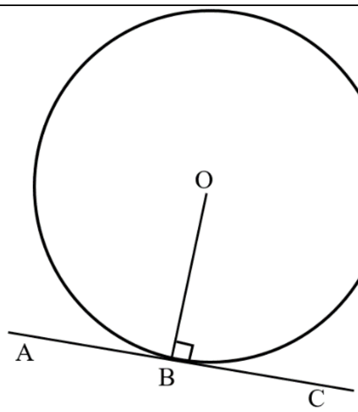
The radius is:

$r^2 = 12 \therefore r = \sqrt{12} = 2\sqrt{3}$

The equation of a tangent to a circle

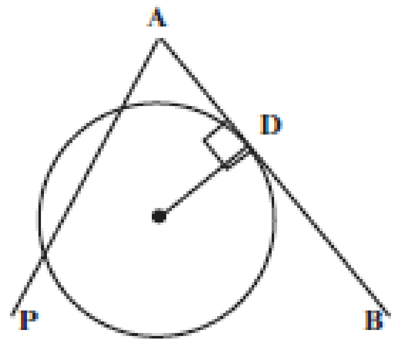
THEOREM 8

A tangent to a circle is perpendicular to the radius at the point of contact.



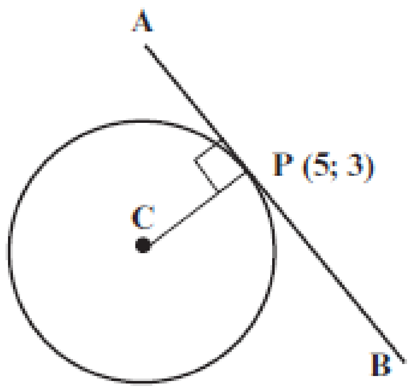
- $m_{RADIUS} \times m_{TANGENT} = -1$ (PERPENDICULAR LINES)

- $m = \frac{y_2 - y_1}{x_2 - x_1}$ (GRADIENT OF A STRAIGHT LINE)
- $y = mx + c$ or $y - y_1 = m(x - x_1)$ (EQUATION OF A STRAIGHT LINE)

A tangent is a straight line which touches a circle at ONE point only. So ADB is a tangent, but AP is NOT a tangent. A tangent to a circle at any point on the circumference is perpendicular to the radius at that point; so $AB \perp CD$.	

WORKED EXAMPLE 5:

Find the equation of the tangent APB which touches a circle centre C with equation $(x-3)^2 + (y+1)^2 = 20$ at $P(5;3)$.

Centre of circle is $C(3;-1)$ so the gradient of the radius CP is: $m_{CP} = \frac{3-(-1)}{5-3} = 2$ radius$\perp$tangent; so, $m_{APB} \times m_{CP} = -1$ and so $m_{APB} = -\frac{1}{2}$ Equation of tangent: $y - y_1 = m(x - x_1)$ $y - 3 = -\frac{1}{2}(x-5)$	

$y-3=-\frac{1}{2}x+\frac{5}{2}$ $y=-\frac{1}{2}x+5\frac{1}{2}$	

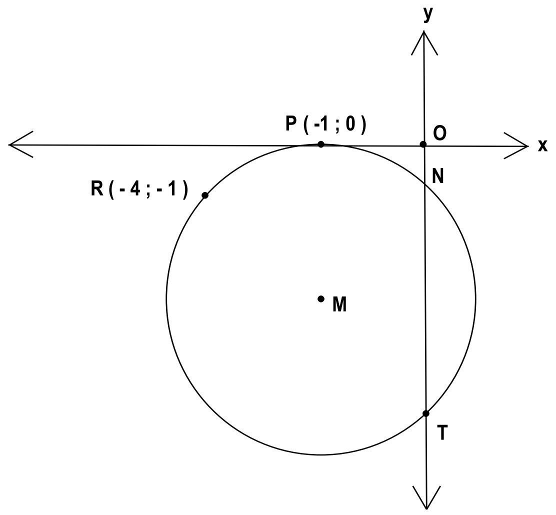
$\therefore$ M lies on AC

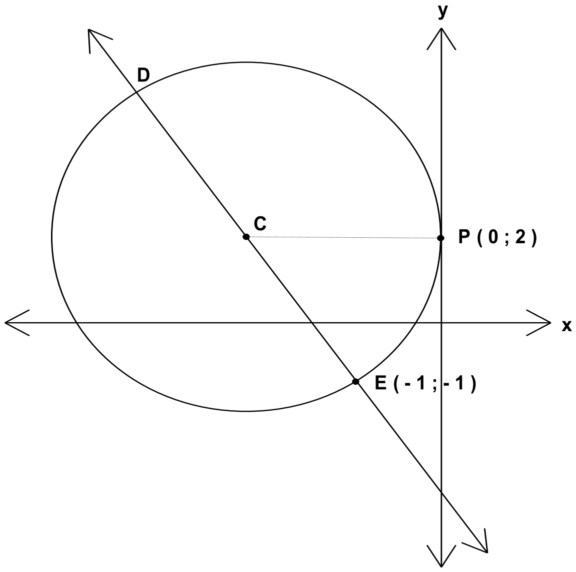
HOMEWORK ACTIVITY

1.	Determine the equation of the circle with centre at the origin and:	
a)	Radius 7	$49=x^2+y^2$
b)	Radius $2\sqrt{7}$	$28=x^2+y^2$
c)	Passing through the point $(2;-1)$	$5=x^2+y^2$
d)	Passing through the point $(-4;-1)$	$17=x^2+y^2$
2.	Determine the radius of each of the following circles:	
a)	$x^2+y^2=1$	$r=1$
b)	$x^2+y^2=100$	$r=10$
c)	$2x^2+2y^2=10$	$r=\sqrt{5}$
3.	Determine the equation of the circle with	
a)	Centre $(-1;1)$ and radius 3 units.	$9=(x+1)^2+(y-1)^2$
b)	Centre $(4;\ -3)$ and passing through the points $(-2;3)$.	$72=(x-4)^2+(y+3)^2$
c)	Centre $(3;0)$ and radius $3\sqrt{2}$ units.	
d)	Centre $(2;3)$ and radius 6.	
e)	Centre $(1;\ -7)$ and radius $\sqrt{5}$.	

f)	Centre$(-3;\ -2)$ and$(0;\ -8)$ a point on the circle.	$45 = (x+3)^2 + (y+2)^2$
g)	Centre $(2; 1)$ and radius $2\sqrt{3}$	
h)	Centre $(3;\ -5)$ and passing through the point $(-1; 1)$	
4.	Determine the coordinates of the centre and the radius of the circle	
a)	$(x-1)^2 + (y-3)^2 = 16$	Centre: $(1; 3)$ Radius: 4
b)	$(x-2)^2 + y^2 = 20$	**Centre: (** **Radius:**
c)	$x^2 + y^2 - 2x - 24 = 0$	**Centre: (** **Radius:**
d)	$x^2 - 4x + y^2 + 2y - 20 = 0$	**Centre: $(2; -1)$** **Radius: 5**
e)	$y^2 + x^2 + 2x - 12 = 0$	**Centre: (** **Radius:**
f)	$x^2 + y^2 - 2x + 4y = 31$	Centre: (1;-2) Radius: $\sqrt{37}$
g)	$(x-5)^2 + (y+1)^2 = 24$	Centre: $(4; 1)$ Radius: $\sqrt{37}$
h)	$x^2 + y^2 - 8x - 2y - 20 = 0$	
5.	Determine the equation of the tangent that touches the circle	

a)	$x^2 + y^2 = 10$ at the point $(-3; 1)$	$y = 3x + 10$
b)	$x^2 + y^2 = 5$ at the point $(1;\ -2)$	
c)	$x^2 + y^2 = 25$ at the point $T\ (3; 4)$	
d)	$4x^2 + 4y^2 = 65$ at the point $(-\frac{1}{2}; 4)$	
6.	The equation of a straight line and a circle are given. For each determine the point (s) of intersection of the line and the circle (if any).	
a)	$y + 2x = 10$ and $x^2 + y^2 = 25$	$(5; 0)$and $(3; 4)$
b)	$3y + x = 5$ and $x^2 + y^2 = 5$	
c)	$x + y = 4$ and $x^2 + y^2 = 8$	
7.	The straight line $y + 3x = 10$ cuts the circle $x^2 + y^2 = 20$ at P and Q. P is in the 1^{st} quadrant.	
a)	Determine the co-ordinates of P and Q.	
b)	Determine the length of the chord PQ	
c)	Determine the equations of the tangents to the circle at P and Q.	
8.	The equation of a circle with centre M is given by: $$x^2 + y^2 + 6y - 7 = 0$$	
a)	Determine the co-ordinates of M.	$M(0; -3)$
b)	Determine, by calculation, whether the straight line $y = x + 1$ is a tangent to the circle or not. Answer: Now because there are 2 answers it means it cuts the circle twice so it is not a tangent to the circle.	

9.	In the diagram below the circle touches the *x*-axis at P(– 1 ; 0) and passes through the point R(– 4; – 1). M is the centre of the circle and the circle cuts the *y*-axis at T and N. Determine: 
a)	The equation of the circle $b = -5$
b)	The equation of the tangent to the circle at R. $(x+1)^2 + (y+5)^2 = 25$
10.	The equation of a circle with centre M and radius *r* is: $$x^2 + y^2 + 4x - 12y + 4 = 0$$
a)	Calculate the co-ordinates of the centre M and the length of the radius *r*.
b)	Without any further calculations write down the co-ordinates of the point(s) where this circle intersects the *x*-axis.
c)	Determine the equation(s) of the tangent(s) to this circle which is parallel to the *y*-axis.

11.	In the diagram below, centre C of the circle lies on the straight line $3x + 4y + 7 = 0$. The straight line cuts the circle at D and E(–1 ; –1). The circle touches the *y*-axis at P(0 ; 2).	
a)	Determine the equation of the circle in the form: $(x-a)^2 + (y-b)^2 = r^2$ $\boxed{(x+5)^2 + (y-2)^2 = 25}$	
b)	Determine the length of diameter DE. $\boxed{D = 10}$	
c)	Determine the equation of the tangent to the circle at D $m_{CD} = \frac{2+1}{-5+1} = -\frac{3}{4} \therefore m_{tangent} = \frac{4}{3}$	