INVERSE FUNCTIONS

OUTCOMES:

1. Definition of Function (including restricted domain)
2. General concept of the *inverse of a function* and how the domain of the function may need to be restricted (in order to obtain a one-to-one function) to ensure that the inverse is a function.
3. Determine and sketch graphs of the inverses of the functions defined by: $\boldsymbol{y = ax + q}$ **(straight line)**, $\boldsymbol{y = ax^2}$ **(parabola) &** $\boldsymbol{y = b^x}$ **(exponential graph).**
4. Understand the Definition of logarithm: $\boldsymbol{y = \log_b x \Leftrightarrow x = b^y}$ **;** $\boldsymbol{b > 0}$ and $\boldsymbol{b \neq 1}$
5. Understand Inverse of exponential is a logarithmic functions.
6. Determine and sketch graph and the inverse of the function defined by: $\boldsymbol{y = b^x}$ for $\boldsymbol{0 < b < 1}$ and $\boldsymbol{b > 1}$.

INTRODUCTION - OVERVIEW OF BASIC CONCEPTS

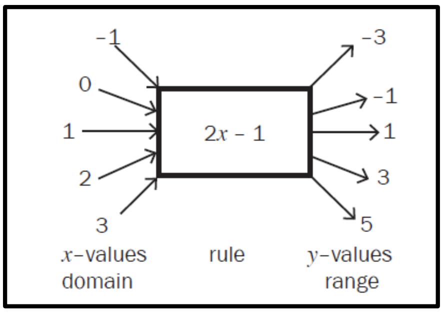
What is a function?

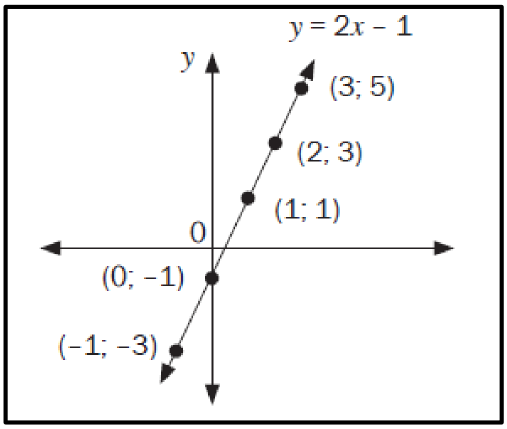
An operator that maps every item in the domain to exactly one item in the range.

Basically:

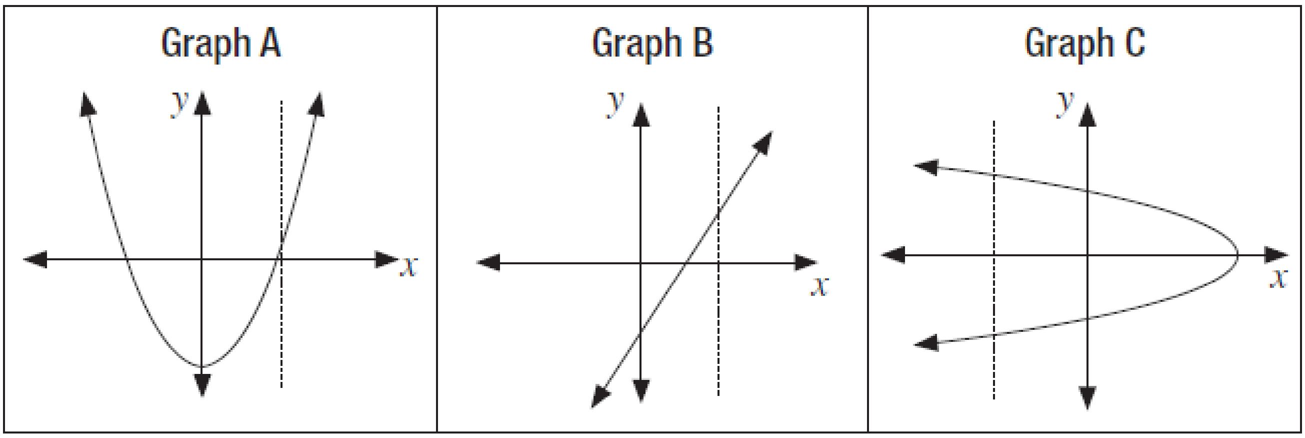
If you are given a set of x-values and you can work out the set of y-values by using a given rule. The x-values are the **input values** and the y-values are the **output values**. In this flow diagram, the rule is; $y = 2x - 1$ and is a function.

So for every x-value, we multiply it by 2 and subtract 1 to find the corresponding y-values. The input values or x-values are the element of the **domain** of this set and the output values or y-values



are the elements of the **range** of this set.	
Look at the graph. For every x-value on this graph there is only one y-value. If the rule or a formula produces only one y-value for each x-value, then we have a **function.**	

A **function** is a relationship between x and y, where for every x-value there is only one y-value. One way to decide whether or not a graph represents a function is to use the **vertical line test**. If any line drawn parallel to the y-axis cuts the graph only once, then the graph represents a function.



Graph A and Graph B are functions. Graph C is NOT a function because the vertical line cuts the graph twice, so for an x-value on the graph, there are two y-values.

FUNCTION NOTATION

We use function notation $f(x)$ to show that each y-value is a function of an x-value. We can also use other letters too, such as $g(x), h(x)$, etc.

So $y = 2x - 1$ can be written as $f(x) = 2x - 1$. The value of $f(x)$ for any x-value can be worked out by substitution:

For example; at $x = -3$ we can find $f(-3) = 2(-3) - 1 = -7$, so the point $(-3; -7)$ lies on the graph of $f(x) = 2x - 1$.

INVERSE FUNCTIONS

1) The inverse of a function takes the y-values (range) of the function to the corresponding x-values (domain) and vice versa, therefore the x and y values are interchanged (swap).
2) The function is reflected along the line $y = x$ to form the inverse.
3) The notation for the inverse of a function is f^{-1}.

WORKED EXAMPLE 1: $\boldsymbol{y = ax + q}$ **(straight line)**

Given $f(x) = 2x + 6$

1. Determine $f^{-1}(x)$

In order to find the inverse of a function, there are two steps:

STEP 1: Swap x and y

$y = 2x + 6$; becomes $x = 2y + 6$

Now we need to make y the subject of the formula.

STEP 2: Make y the subject of the formula

$x = 2y + 6$

$2y = x - 6$

$$y = \frac{1}{2}x - 3$$

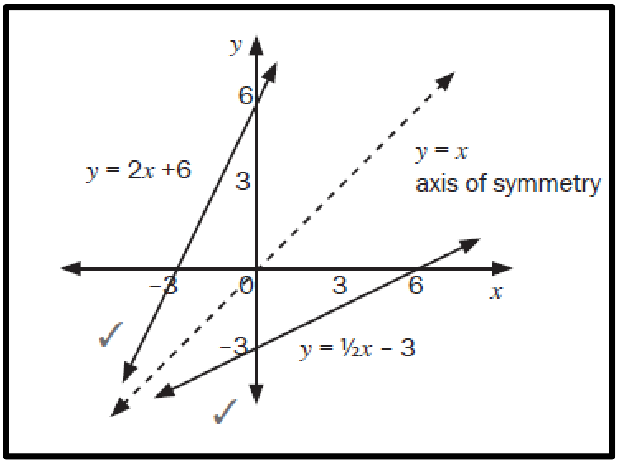
We can say that the inverse function $f^{-1}(x) = \frac{1}{2}x - 3$

2. Sketch the graphs of $f(x)$, f^{-1} and $y = x$ and the same set of axis.

$f(x) = 2x + 6$	$f^{-1}(x) = \frac{1}{2}x - 3$	$y = x$
$y = 2x + 6$ $y = 0, x = -3$ $x = 0, y = 6$	$y = \frac{1}{2}x - 3$ $y = 0, x = 6$	

x	-3	0	3
y	-3	0	3

	$x = 0, y = -3$	



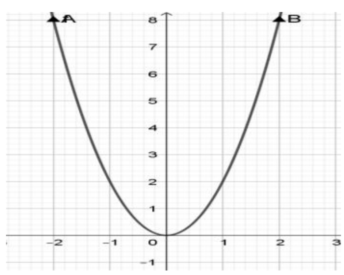
IMPORTANT NOTES:

- ✓ Every point on the function has the same coordinates as the corresponding point on the inverse function, except that they are swapped around.
- ✓ Example: $(-3; 0)$ on the function is reflected to become $(0; -3)$ on the inverse function.
- ✓ Any point $(a; b)$ on the function becomes the point $(b; a)$ on the inverse.
- ✓ To find the equation of an inverse function algebraically, we interchange (swap) x and y and then make y subject formula.
- ✓ To draw the graph of the inverse function, we reflect the original graph about the line $y = x$, the axis of symmetry of the two graphs.

WORKED EXAMPLE 2: $y = ax^2$ (parabola)

1. Sketch $f(x) = 2x^2$

x	-2	-1	0	1	2
y					



2. Determine the inverse of $f(x)$

$y = 2x^2$

$x = 2y^2$

$\frac{x}{2} = y^2$

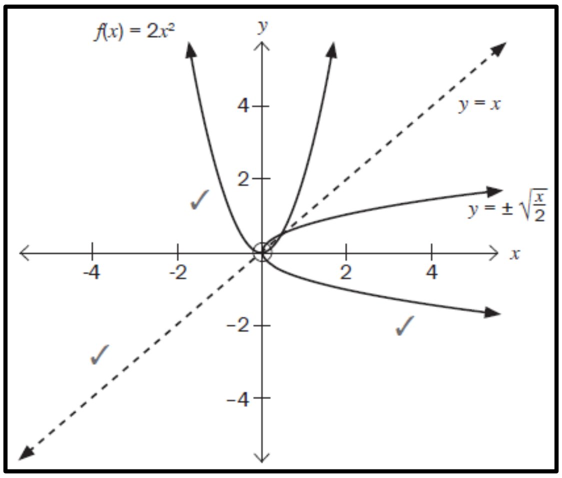
$y = \pm\sqrt{\frac{x}{2}}$

IMPORTANT NOTES:

- ✓ This is NOT a function. Check it with a vertical line test. There are two y-values for one x-value.
- ✓ Not all inverses of functions are also functions. If an inverse is NOT a function, then we can restrict the DOMAIN of the function in order for the inverse to be a function.
- ✓ To make the inverse a function, we need to choose a set of x-values in the function and work only with those. We call this 'restricting the domain'

3. Sketch $f^{-1}(x)$ and $y = x$ on the same axes as $f(x)$

x	0	2	4
f^{-1}	0	±1	$\pm1{,}41$



<u>The logarithmic function</u>

- $y = \log_x a$ is a logarithmic function with $a =$ log number, $x =$log base
- $y = \log_x a$ READS "y is equal to log a base x"
- The logarithmic function is only defined if $a > 0$; $a \neq 1$ and $x > 0$
- An exponential equation can be written as a logarithmic equation and vice versa

IMPORTANT NOTE:

The inverse of the exponential function $y = a^x$ is $x = a^y$. In order to make y the subject of the formula, $x = a^y$, we use the LOG function

$y = \log_a x$ in the inverse of $y = a^x$

WORKED EXAMPLE 3: $y = b^x$ (exponential graph)

Given: $f(x) = 2^x$

1. Determine f^{-1} in the form $y =$..

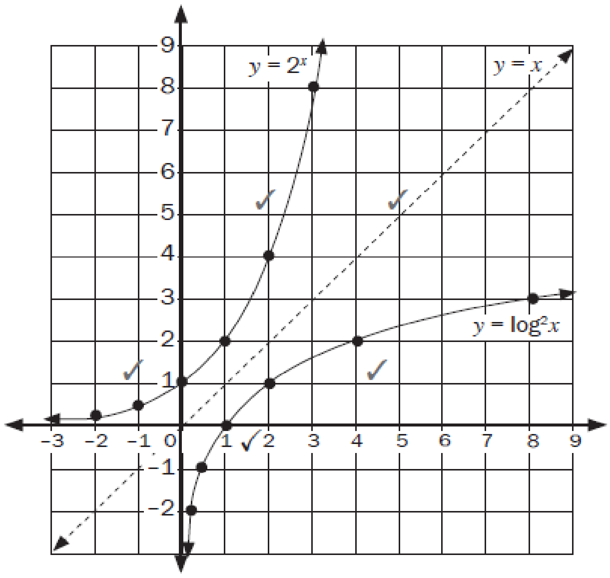
The inverse of the exponential function $y = 2^x$ is $x = 2^y$ which can be written as $y = \log_2 x$.

2. Sketch the graphs of $f(x), f^{-1}(x)$ and $y = x$ on the same set of axes.

x	-2	-1	0	1	2	3
$y = 2^x$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$y = \log_2 x$	-2	-1	0	1	2	3

x	-3	0	6
$y = x$	-3	0	6



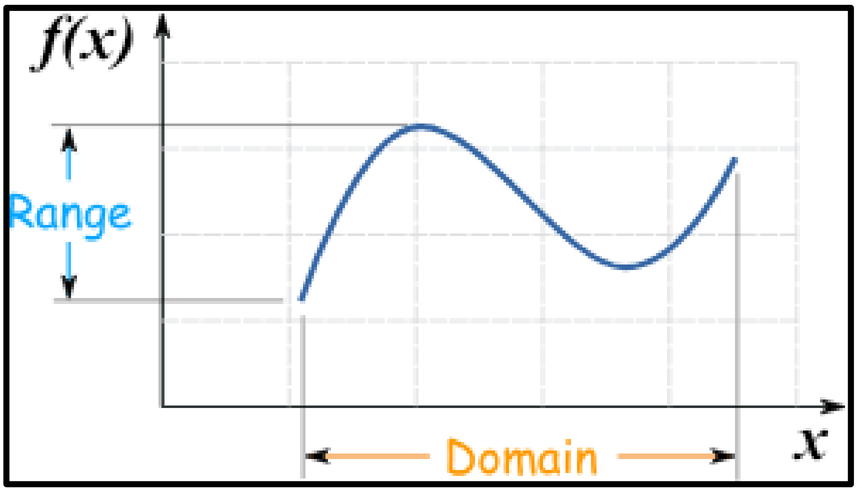
The graph of $y = \log_2 x$ is a **reflection** about the $y = x$ axis of the exponential graph of $y = 2^x$

3. Write the domain and range of $f(x)$ and $f^{-1}(x)$

DOMAIN AND RANGE OF PARABOLA

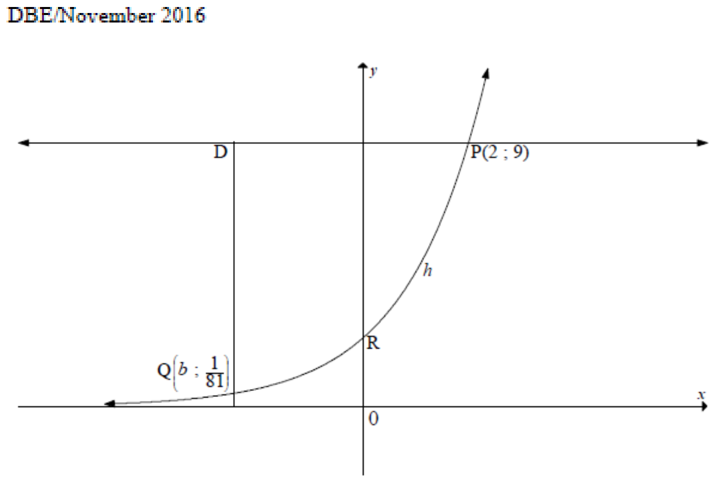
Domain is the set of x-values for which the graph or function has been defined.

Range is the set of y-values for which the graph of function has been defined.

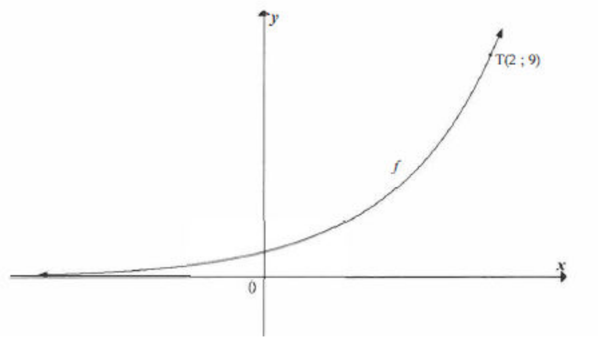
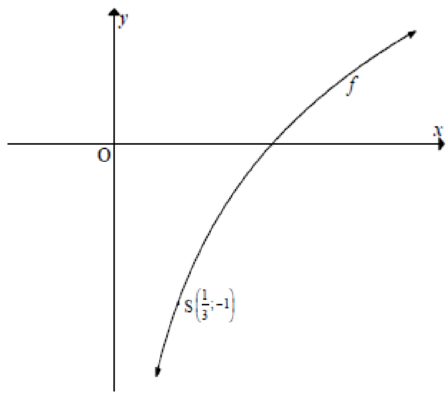


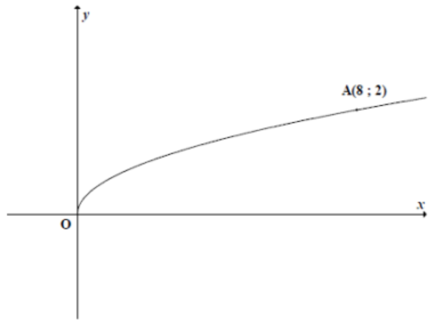
	Domain	Range
$f(x)$	$xE\mathbb{R}$	$y > 0$
$f^{-1}(x)$	$x > 0$	$yE\mathbb{R}$

HOMEWORK ACTIVITY

1.	Sketched is the graph of f, the inverse of a restricted parabola. The point A(8 ; 2) lies on the graph of f.	
1.1	Write down the equation of the asymptote of *h*.	
1.2	Determine the coordinates of R.	
1.3	Calculate the value of *a*.	
2.	Given: $f(x) = -x + 3$ and $g(x) = 3^x$	
2.1	On the same set of axes, sketch the graphs of *f* and *g*, clearly showing ALL intercepts with the axes.	
2.2	Write down the equation of $g^{-1}(x)$, the inverse of *g*, in the form $y =$..	
3.	Given: $f(x) = 2^{-x} + 1$	
3.1	Determine the coordinates of the y-intercept *of* f	
3.2	Sketch the graph of f, clearly indicating ALL intercepts with the axes as well as any asymptotes.	
3.3	If $h(x) = 3f(x)$, write down an equation of the asymptote of *h*.	
4.	Given: $f(x) = \frac{1}{4}x^2, x \leq 0$	

4.1	Determine the equation of f^{-1} in the form $f^{-1}(x)$ = . .
4.2	On the same system of axes, sketch the graphs of f and f^{-1}. Indicate clearly the intercepts with the axes, as well as another point on the graph of each *of* f and f^{-1}.
4.3	Is f^{-1} a function? Give a reason for your answer.
5.	Given: $f(x) = \left(\frac{1}{4}\right)^{x}$
5.1	Determine the value of $f(-2)$.
5.2	Write down the equation of $f^{-1}(x)$ in the form $y =$..
5.3	Andrew has no idea how to draw the graph of f^{-1}. Explain to Andrew how he may use the graph of f to draw the graph of f^{-1}.
5.4	Hence or otherwise, sketch the graph off^{-1} in your ANSWER BOOK. Clearly indicate ALL intercepts with the axes.
5.5	Write down the domain of f^{-1}.
5.6	For which values of *x* will $f^{-1}(x) \geq -2$?
5.7	Given: $q = \log_{\frac{1}{4}} \frac{1}{2}$
5.7.1	Determine the value of *q*.
5.7.2	Hence or otherwise, determine the coordinates of the point of intersection of f and f^{-1}.

6.	The graph of $\boldsymbol{f(x) = a^x, a > 1}$ is shown alongside. $\boldsymbol{T(2; 9)}$ lies on $\boldsymbol{f}$.	
6.1	Calculate the value of a.	
6.2	Determine the equation of $g(x)$ if $g(x) = f(-x)$.	
6.3	Determine the value(s) of x for which $f^{-1}(x) \geq 2$.	
6.4	Is the inverse of f a function? Explain your answer.	
7.	Given: $f(x) = \log_a x$ where $a > 0$. $S\left(\frac{1}{3}; -1\right)$ is a point on the graph of f.	
7.1	Prove that $a = 3$.	
7.2	Write down the equation of h, the inverse of f, in the form $y =$..	
7.3	If $g(x) = -f(x)$, determine the equation of g.	
7.4	Write down the domain of g.	
7.5	Determine the values of x for which $f(x) \geq -3$.	
8.	The graph of g is defined by the equation $g(x) = \sqrt{ax}$. The point $(8; 4)$ lies on g.	
8.1	Calculate the value of a.	

8.2	If $g(x) > 0$, for which values of x will g be defined?
8.3	Determine the range of g.
8.4	Write down the equation of g^{-1}, the inverse of g, in the form $y =$..
8.5	If $h(x) = x - 4$ is drawn, determine ALGEBRAICALLY the point(s) of intersection of h and g.
8.6	Hence, or otherwise, determine the values of x for which $g(x) > h(x)$.
9.	Sketched is the graph of , the inverse of a restricted parabola. The point A(8 ; 2) lies on the graph of *f*. (WCED Sep 2015) 
9.1	Determine the equation of *f* in the form *y* =
9.2	Hence, write down the equation of f^{-1} in the form *y* = …
9.3	Give the coordinates of the turning point of $g(x) = f^{-1}(x + 3) - 1$
10.	Given $f(x) = \left(\frac{1}{5}\right)^x$ **EC Sep 2014**
10.1	Is *f* an increasing or decreasing function? Give a reason for your answer.
10.2	Write down the range *g* of if $g(x) = f(x) - 2$.
10.3	Write down the equation of f^{-1}, the inverse of *f*, in the form *y* = …
10.4	Draw a neat sketch of f^{-1}. Clearly show all intercepts with the axes.

10.5	For which value(s) of is x is $f^{-1}(x) \geq -1$
11.	Given: $f(x) = -3x^2$
11.1	Write down the equation for the inverse function in the form $y =$..
11.2	Determine the domain and range of $f(x)$ and $f^{-1}(x)$
11.3	Determine the points of intersection of $f(x)$ and $f^{-1}(x)$
12.	Given: $g(x) = 3x + 2$
12.1	Find $g^{-1}(x)$
12.2	Sketch g, g^{-1} and the line $y = x$ on the same set of axes.
13	Given: $f(x) = 2^x$
13.1	Determine f^{-1} in the form $y =$..
13.2	Sketch the graphs of $f(x), f^{-1}(x)$ and $y = x$ on the same set og axes.
13.3	Write the domain and range of $f(x)$ and $f^{-1}(x)$.