# CALCULUS LESSON 1

## OUTCOMES:

1. Using the definition (first principle) to find the derivative, $f'(x)$ where *a*, *b* and *c* constants.
2. Find the derivative by using the rules.

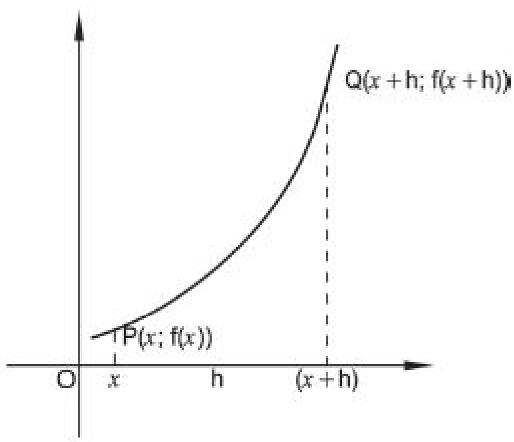
## INTRODUCTION - OVERVIEW OF BASIC CONCEPTS

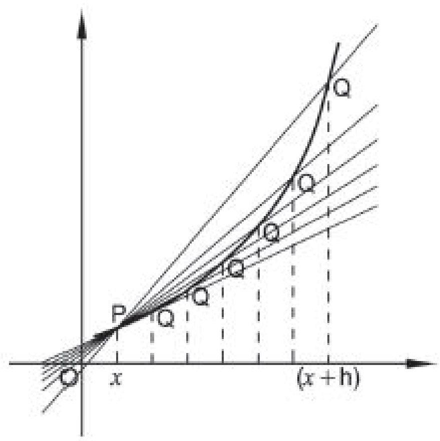
Calculus is one of the most important and powerful branches of Mathematics, that originated mainly from attempts to solve two problems:

1. To define and find the equation of a tangent to a curve.
2. To define and find the area of a region bound by a curve.

### Gradient and average gradient

The gradient or slope of a line gives us an indication of the steepness of the line relative to the horizontal axis.

| | |
|---|---|
| The gradient of a straight line can be calculated using:<br>$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{change in y}}{\text{change in } x}$<br>The average gradient between P and Q is:<br>$= \frac{f(x+h) - f(x)}{x + h - x}$<br>$= \frac{f(x+h) - f(x)}{h}$<br>What if we want the gradient at P? The gradient is $\frac{0}{0}$. What is that? We need to make a plan? As the two points move closer and closer to each other, the value of h gets closer and closer to zero |  |

| $h \to 0$ | |
|---|---|
| The closer $h$ gets to zero, the closer we are to the gradient of a curve at a point or the gradient of a tangent. This is known as the gradient function. It is:<br>• The gradient of a curve at a point<br>• The gradient of a tangent<br>• The instantaneous rate of change<br>• The derivative.<br>We need a new notation: we use $\frac{lim}{h \to 0}$ |  |

This means the distance between the two points is very, very, very small (very close to 0) the gradient $= \frac{f(x+h)-f(x)}{h}$

The gradient of the tangent $\lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h}$ (instantaneous gradient of the derivative).

When we use $\lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h}$ to differentiate we say we are differentiating from **FIRST PRINCIPLES**.

**WORKED EXAMPLE 1:**

Determine $f'(x)$ from first principles if $f(x) = -3x^2$

| | |
|---|---|
| $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$<br>**Substitute into formula**<br>$f'(x) = \lim\limits_{h \to 0} \dfrac{-3x^2 - 6xh - 3h^2 - (-3x^2)}{h}$<br>$f'(x) = \lim\limits_{h \to 0} \dfrac{-3x^2 - 6xh - 3h^2 + 3x^2}{h}$ | $f(x) = -3x^2$<br>**To get $f(x+h)$ we replace $x$ with $(x+h)$ and simplify**<br>$f(x+h) = -3(x+h)^2$<br>$f(x+h) = -3(x^2 + 2xh + h^2)$<br>$f(x+h) = -3x^2 - 6xh - 3h^2$ |

$$f'(x) = \lim_{h\to 0}\frac{-6xh - 3h^2}{h}$$

**Take out a common factor $h$ so you can cancel it with the $h$ in the denominator.**

$$f'(x) = \lim_{h\to 0}\frac{h(-6x - 3h)}{h}$$

$$f'(x) = \lim_{h\to 0}(-6x - 3h)$$

**As $h$ goes to 0; $-6x - 3h$ goes to $-6x$.**

$$\therefore f'(x) = -6x$$

**WORKED EXAMPLE 2:**

Determine $f'(x)$ from first principles if $f(x) = \frac{2}{x}$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

**Substitute into formula**

$$f'(x) = \lim_{h\to 0}\frac{\frac{2}{x+h} - \frac{2}{x}}{h}$$

**Use $x(x+h)$ as your LCD**

$$f'(x) = \lim_{h\to 0}\frac{\frac{2x - 2(x+h)}{x(x+h)}}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{\frac{2x - 2x - 2h}{x(x+h)}}{h} \times \frac{\frac{x(x+h)}{1}}{\frac{x(x+h)}{1}}$$

$$f'(x) = \lim_{h\to 0}\frac{-2h}{hx(x+h)}$$

$$f'(x) = \lim_{h\to 0}\frac{-2}{x(x+h)}$$

**As $h$ goes to 0; $x + h$ goes to $x$**

$$f(x) = \frac{2}{x}$$

**To get $f(x+h)$ we replace $x$ with $(x+h)$**

$$f(x+h) = \frac{2}{x+h}$$

| | |
|---|---|
| $f'(x) = \frac{-2}{x(x)}$<br>$f'(x) = \frac{-2}{x^2}$ | |

## The rules of differentiation

You can find any derivative from first principles, but there are some quick rules to find the derivative. **Unless a question asks you to use the definition or to differentiate from first principles,** it is easier to use the rules. You need to know and be able to use the following rules for differentiating:

$$\boldsymbol{f(x) = x^n, then, f'(x) = nx^{n-1}}$$

**The derivative of a constant is always 0**

1. Before you use differentiation you might need to **simplify (expand brackets) or change** the format of the expressions.
2. Rewrite terms which are **square roots, cube roots of any other roots** as **exponentials** so you can use the differentiation rules.
3. Rewrite terms which are in **FRACTIONS** form, where $x$ is part of the denominator, $\frac{1}{x^n} = x^{-n}$ so you can use the differentiation rules.

**WORKED EXAMPLE 3:**

Find the derivative of the following:

| | |
|---|---|
| 3.1 | Determine $f'(x)$ if: $f(x) = 3x^2 - 4$<br>$f'(x) = 2 \times 3x^{2-1} - 0$<br>$f'(x) = 6x$ |
| 3.2 | Determine $f'(x)$ if: $f(x) = 7x^5 - 3x^3 + 5x^2$<br>$f'(x) = 5 \times 7x^{5-1} - 3 \times 3x^{3-1} + 2 \times 5x^{2-1}$ |

| | |
|---|---|
| | $f'(x) = 35x^4 - 9x^2 + 10x$ |
| 3.3 | Determine $f'(x)$if: $f(x) = (3x+2)(x-5)$<br>**You do NOT have a rule for differentiating a product, so you first need to simplify before you can differentiate.**<br>$f(x) = 3x^2 - 15x + 2x - 10 = 3x^2 - 13x - 10$<br>$\therefore f'(x) = 6x - 13$ |
| 3.4 | Determine $\frac{dy}{dx}$ if $y = 2(2x-1)^2$<br>$y = 2(4x^2 - 4x + 1)$<br>$y = 2(4x^2 - 4x + 1)$<br>$y = 8x^2 - 8x + 2$<br>$\frac{dy}{dx} = 16x - 8$ |
| 3.5 | Determine $f'(x)$if: $f(x) = \sqrt{x}$<br>**First rewrite in exponential form**<br>$f(x) = x^{\frac{1}{2}}$<br>$\therefore f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$ |
| 3.6 | Determine $f'(x)$if: $f(x) = \frac{2}{x}$<br>$f(x) = \frac{2}{x^1} = 2x^{-1}$<br>$\therefore f'(x) = -1 \times 2x^{-1-1}$<br>$\therefore f'(x) = -2x^{-2}$ |
| 3.7 | $D_x\left[5\sqrt{x} - \frac{5}{x^2}\right]$ |

| | |
|---|---|
| | $= D_x\left[5x^{\frac{1}{2}} - 5x^{-2}\right]$<br>$= \frac{5}{2}x^{-\frac{1}{2}} + 10x^{-3}$ |

## HOMEWORK ACTIVITY

| | |
|---|---|
| 1. | Determine $f'(x)$ from first principles if: |
| 1.1 | $f(x) = 1 - 3x^2$ |
| 1.2 | $f(x) = 2x^2$ (March 2012) |
| 1.3 | $f(x) = 1 - 2x^2$ (November 2014) |
| 1.4 | $f(x) = \frac{4}{x}$ |
| 1.5 | $f(x) = -\frac{2}{x}$ |
| 1.6 | $f(x) = -x^3$ |
| 1.7 | $f(x) = \frac{1}{2}x^2 + x - 2$ |
| 1.8 | $f(x) = -x^2 + 4x - 1$ |
| 1.9 | $f(x) = 5x^2 - 4x + 2$ |
| 2. | Find the derivative of the following: |
| 2.1 | $f(x) = 1 - 3x^2$ |
| 2.2 | $f(x) = 2x^2$ |
| 2.3 | $f(x) = 1 - 2x^2$ |
| 2.4 | $f(x) = 4x - 3$ |

| | |
|---|---|
| 2.5 | $f(x) = -5x^2 + x$ |
| 2.6 | $f(x) = \frac{4}{x}$ |
| 2.7 | $f(x) = -\frac{2}{x}$ |
| 2.8 | $f(x) = -x^3$ |
| 2.9 | $f(x) = \frac{1}{2}x^2 + x - 2$ |
| 2.10 | $f(x) = -x^2 + 4x - 1$ |
| 2.11 | Calculate the derivative of: (FEB – MARCH 2017)<br>a) $y = 3x^2 + 10x$<br>b) $f(x) = \left(x - \frac{3}{x}\right)^2$ |
| 2.12 | Determine $\frac{dy}{dx}$ if: (NOV 2016)<br>a) $y = 2x^5 + \frac{4}{x^3}$<br>b) $y = \left(\sqrt{x} - x^2\right)^2$ |
| 2.13 | Determine the derivative of: $g(x) = 5x^2 - \frac{2x}{x^3}$ (FEB – MARCH 17) |
| 2.14 | Determine the derivative of: $f(x) = 2x^2 + \frac{1}{2}x^4 - 3$ (NOV 2014) |
| 2.15 | Determine $\frac{dy}{dx}$ if $y = \sqrt{x^3} - \frac{5}{x^3}$ (NOV 2016) |
| 2.16 | Determine the derivative of: (NOV 2015)<br>a) $\frac{dy}{dx}$ if $y = \left(x^2 - \frac{1}{x^2}\right)^2$ |

| | |
|---|---|
| | b) $D_x = (\frac{x^3-1}{x-1})$ |
| 2.17 | Determine: (DBE 2015)<br>a) $\frac{dy}{dx}$ if $y = 5x^2 + 5x + 2$<br>b) $D_x\left[\sqrt[3]{x^2} - \frac{1}{2}x\right]$ |
| 2.18 | Determine $\frac{dy}{dx}$ if $y = 2x^{-4} - \frac{x}{5}$ |
| 2.19 | Differentiate: FEB – MARCH 2015<br>a) $f(x) = -3x^2 + 5\sqrt{x}$<br>b) $p(x) = \left(\frac{1}{x^3} + 4x\right)^2$ |