CALCULUS LESSON 2

OUTCOMES:

1. Find equations of tangents to graphs of functions.
2. Understand intuitively that $f\,'(a)$ is the gradient of the tangent to the graph of f at the point with x -coordinate a.
3. Factorise third-degree polynomials. Apply factor and remainder theorem to polynomials of degree at most 3.

INTRODUCTION - OVERVIEW OF BASIC CONCEPTS

THE DERIVATIVE HAS MANY USES.

It can be used to:

- Find the gradient of the equation of a tangent line
- Identify stationary points on a graph
- Find a maximum or minimum value
- Describe rates of change
- Draw graphs of cubic functions

FINDING THE EQUATION OF A TANGENT LINE

The slope of the tangent line to the graph at a point is equal to the derivative of the function at that point. So, to find the equation of the tangent line to $f(x)$ at $x = a$, we must:

1. Find the derivative $f'(x)$
2. Work out the derivative at $x = a$ → i.e calculate $f\,'(a)$ to get the
3. gradient of the tangent line.
4. Calculate the y value at $x = a$ →i.e calculate $f(a)$.
5. The tangent line is a straight line.
6. We can find the equation of a straight line using $y - y_1 = m(x - x_1)$, if we know the gradient m and a point $(x_1; y_1)$ on the line.

WORKED EXAMPLE 1:

Find the equation of the tangent to the function $f(x) = x^3 + 2x + 4$ at the point where $x = 1$.

STEPS:

1. Find the derivative
2. Find the gradient of the tangent at $x = 1$, by substituting $x = 1$ in the derivative.
3. Calculate the y-value at $x = 1$, by substituting $x = 1$ into the **original equation**.
4. Use $y - y_1 = m(x - x_1)$ to give the equation of the tangent line

$m = f'(x) = 3x^2 + 2$

$m = f'(1) = 3(1)^2 + 2$

$m = 5$

$y = f(1) = x^3 + 2x + 4$

$y = f(1) = (1)^3 + 2(1) + 4$

$y = 7$

POINT$= (1\,;7)$

The equation of the tangent line: $y - y_1 = m(x - x_1)$

$y - 7 = 5(x - 1)$

$y - 7 = 5x - 5$

$y = 5x + 2$

So the equation of the tangent at $x = 1$ is $y = 5x + 2$

DRAWING THE GRAPH OF A CUBIC POLYNOMIAL

A cubic polynomial is a function of the form $f(x) = ax^3 + bx^2 + cx + d$ and we can represent it with a graph, we need to work out the characteristics of the graph.

We also need to know how to solve equations in the third degree, so that we can work out the x- and y- intercepts of the graph. We use the factor theorem to factorise an expression. First, we use the theorem to find one factor by the trial and error method. Then, we factorise completely through long division or **SYNTHETIC DIVISION.**

WORKED EXAMPLE 2:

Factorise and solve for x: $f(x) = x^3 - x^2 - 5x - 3$

The factor theorem states:

If $\boldsymbol{f(k) = 0}$, then $\boldsymbol{x - k}$ is a factor of $\boldsymbol{f(k)}$

Basically: If we are going to replace x with a number in $f(x) = x^3 - x^2 - 5x - 3$ and the answer is equal to 0.

So if $f(x) = x^3 - x^2 - 5x - 3$, we want to find an x-value that makes $f(x) = 0$. $f(x)$ has a constant value of -3.

If this expression can be factorised, we need to use the factors of -3.

The factors are: $-3 \quad ; \quad 3 \quad ; \quad 1 \quad ; \quad -1$

By trial and error; test these factors to find which value of x gives $f(x) = 0$.

If $x = -3$, then $f(-3) = x^3 - x^2 - 5x - 3 = -24$

If $x = -1$, then $f(-1) = x^3 - x^2 - 5x - 3 = 0$

$\therefore x - (k)$ is factor of $f(x)$

$\therefore x - (-1)$ is factor of $f(x)$

$\therefore x + 1$ is a factor of $f(x) = x^3 - x^2 - 5x - 3$

We will use $x + 1$ to find the other factors.

STEP 1: Write down the coefficients of the x - terms and the constant term.

STEP 2 : Place $x = -1$ of the left, this is the value you use to make $f(x) = 0$, that number must be written on your left

Step 3: Use the the same coefficient of the first term and write it on op of the first term.

Step 4: multiply $1 \times -1 = -1$ write this number under the 2nd term and add the second column and place this number on top.

Step 5: multiply $-2 \times -1 = 2$ and write this number under the 3rd term and add the third column and place this number on top.

Step 6: Multiply $-3 \times -1 = 3$, put this number under the 4th term and add the fourth column and place this number on top.

You know you are correct when the final sum is 0. These numbers form the coefficients of the trinomial:

$$\begin{array}{r|l} & \mathbf{1 - 2 - 3 + 0} \\ \hline \mathbf{-1} & \mathbf{1 - 1 - 5 - 3} \\ & \quad \mathbf{-1 + 2 + 3} \end{array}$$

$$a = 1\,; b = -2\,; c = -3$$

$$\therefore 1x^2 - 2x - 3$$

We found the first factor already by using the factor and remainder theorem.

Factorise the answer further by factorising the trinomial.

$= (x + 1)(x^2 - 2x - 3) = (x + 1)(x - 3)(x + 1)$

Determine the three solutions, if $(x + 1)(x - 3)(x + 1) = 0$

Then:

$x + 1 = 0$	or	$x - 3 = 0$	or	$x + 1 = 0$

$x = -1$		$x = 3$		$x = -1$

These are the x-intercepts of a cubic graph with the equation:

$$f(x) = x^3 - x^2 - 5x - 3$$

WORKED EXAMPLE 3:

$f(x) = -2x^3 + x^2 - 4x + 2$

a) Prove that $2x - 1$ is a factor of $f(x)$.

If you are going to replace $x = \frac{1}{2}$ in $f(x)$ and if the answer is equal to 0, then $2x - 1$ is a factor of $f(x)$.

$f\left(\frac{1}{2}\right) = -2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^2 - 4\left(\frac{1}{2}\right) + 2 = 0$

Hence $(2x - 1)$ MUST be a factor because $f\left(\frac{1}{2}\right) = 0$

b) Hence factorise completely

$x - \left(\frac{1}{2}\right)$ is factor of $f(x)$

$$\begin{array}{c|l} \frac{1}{2} & -2 + 1 - 4 + 2 \\ & \mathbf{-2} - 1 + 0 - 2 \\ & \qquad \mathbf{0 - 4} \ \ \mathbf{+0} \end{array}$$

$\therefore f(x) = (2x - 1)(-2x^2 + 0x - 4)$

$\therefore f(x) = (2x - 1)(-2x^2 - 4)$

WORKED EXAMPLE 4:

$f(x) = -x^3 + kx^2 - x + 3$. Determine the value of k if $(3 - x)$ is a factor of $f(x)$.

If $-x + 3$ is a factor of $f(x)$, then according to the factor theorem $f(3) = 0$.

$\therefore f(3) = -(3)^3 + k(3)^2 - 3 + 3 = 0$

$\therefore -27 + 9k = 0$

$\therefore k = 3$

HOMEWORK ACTIVITY

1.	Find the equation of the tangent to the curve at the given value of x.
1.1	$f(x) = -x^2 + 2x + 3$ at $x = 2$. $\boxed{y = -2x + 7}$
1.2	$y = x^2 - x - 6;$ at $x = 0$
1.3	$y = -3x^2 + 2x + 1;$ at $x = 1$ $\boxed{y = -4x + 4}$
1.4	$y = -\frac{6}{x};$ at $x = -2$
1.5	$y = x^3 - 4x + 2;$ at $x = 2$
1.6	$y = -2x^3 - 6x^2 + 4;$ at $x = -1$
2	$g(x) = -8x + 20$ is a tangent to $f(x) = x^3 + ax^2 + bx + 18$ at $x = 1$. Calculate the values of a and b.
3	Show that the point $(2; 8)$ lies on the curve $y = x^3$ and find the equation of the tangent to the curve through this point.
4	Given: $p(x) = x^3 + 2x$ Show, using relevant calculations, why it is NOT possible for a tangent drawn to the graph of p to have negative gradient.
5	Find the equation of the tangent line to$y = 1 - 3x^2$ at $x = -1$.
6	If$f(x) = 2 - 3x - 2x^2$, at what point on the curve will the gradient of the tangent be 13?
7	At which point on the curve will the tangent be parallel to the line $y = -6x + 10$, if the curve has the equation $y = x^2 - 4x + 2$.
8	$f(x) = x^3 + x^2 + ax - b$. Find the values of a and b if the gradient of the tangent at$(1; -2)$ is 4.

9	The distance a rocket travels in the initial stages of lift-off is calculated using the formula $d(t) = t^3$. Calculate the speed of the rocket 10 seconds after lift-off.
10	The value of an investment is calculated using $V(t) = \sqrt{t^3}$, where V is the value and t is the time in years. Calculate the growth **rate** of the investment after 9 years.
11	Use the **SYNTHETIC METHOD** to determine the x-intercepts of a cubic graph.
11.1	$f(x) = 2x^3 + x^2 - 7x - 6$
11.2	$f(x) = x^3 + 3x^2 - 6x - 8$ $(x+1)(x+4)(x-2)$
11.3	$f(x) = -x^3 - x^2 + x + 10$
11.4	$f(x) = -2x^3 + 5x^2 + 4x - 3$ $(x-3)(2x-1)(x+1)$
11.5	$f(x) = x^3 - 4x^2 - 11x + 30$
11.6	$f(x) = -x^3 + 3x^2 - 2$
11.7	$f(x) = 2x^3 - x^2 - 8x + 4$
11.8	$f(x) = -2x^3 - 3x^2 + 12x + 20$
11.9	$f(x) = x^3 - x^2 - 16x + 16$
11.10	$f(x) = x^3 - 5x^2 - 8x + 12$
12	Prove that $x + 2$ is a factor of $x^3 + 9x^2 + 26x + 24$ and hence factorise the expression completely. $(x+2)(x+4)(x+3)$
13	Prove that $x + 3$ is a factor of the expression $x^3 + 2x^2 - 5x - 6$ and hence factorise the expression completely.
14	Prove that $f(x) = 6x^3 + 11x^2 - 4x - 4$ is divisible by $2x + 1$,

	and hence factorise $f(x)$.
15	$f(x) = 8x^3 + 4x^2 - 2x - 1$ and $g(x) = 2x + 1$. Prove that $f(x)$ is divisible by $g(x)$ and hence factorise $f(x)$ completely.
16	Prove that $(1 - 2x)$ is a factor of $f(x) = -2x^3 + x^2 - 2x + 1$
17	Determine the value of p if $(2 - x)$ is a factor of $x^3 + px + 6$.
18	Determine the values of a and b if $x - 1$ and $x + 1$ are factors of $ax^3 + bx^2 + 5x - 5$
19	Determine the values of a and b if $x + 1$ and $x - 2$ are factors of $x^3 + ax^2 + bx + 8$.