CALCULUS LESSON 3

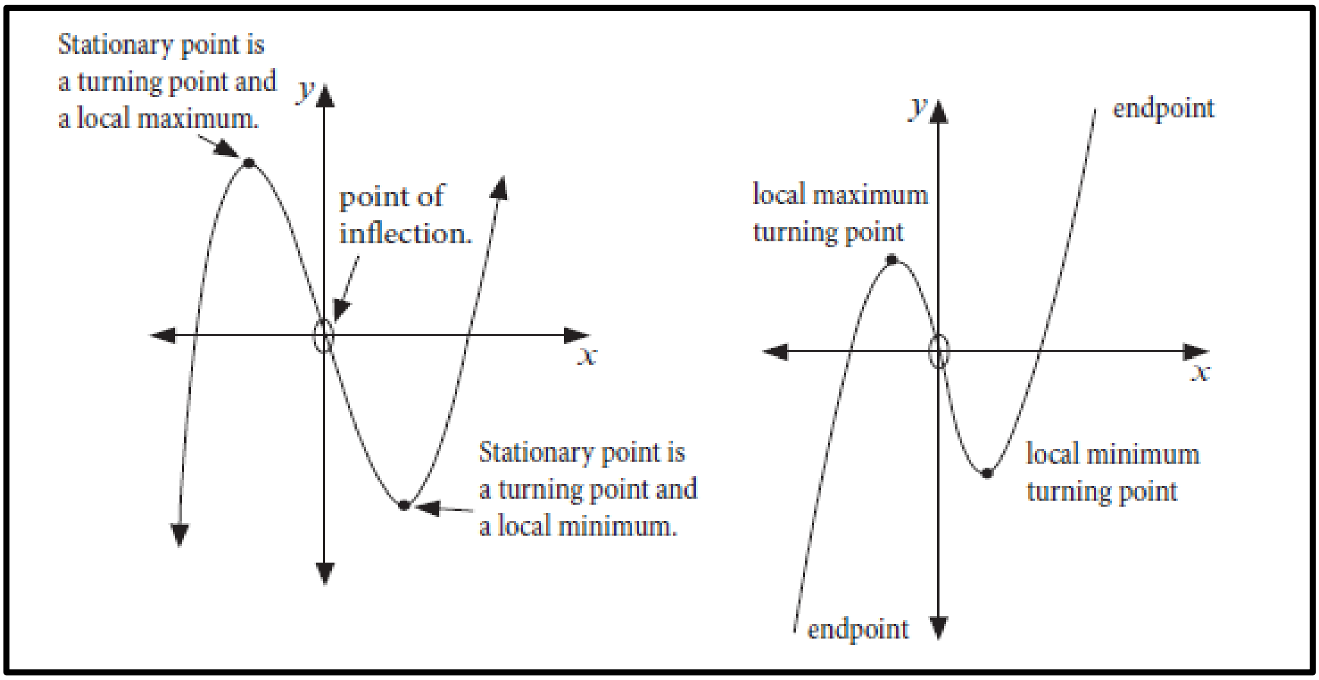
OUTCOMES:

1. Introduce the second derivative of $f''(x)$ and how it determines the concavity of a function.
2. Sketch graphs of cubic polynomial functions using differentiation to determine the Coordinates of stationary points, and points of inflection (where concavity changes). Also, determine the x - intercepts of the graph using the factor theorem and other techniques.

INTRODUCTION - OVERVIEW OF BASIC CONCEPTS

Stationary points on a graph are points where the gradient of the graph is 0. This is at points where the direction of the curve of the graph changes. On a cubic function, the stationary points are at local maximum or minimum turning point. There are also situations where a point of inflection can also be a stationery point as indicated on the figures below.

NOTE: A point of inflection is not always a stationary point.



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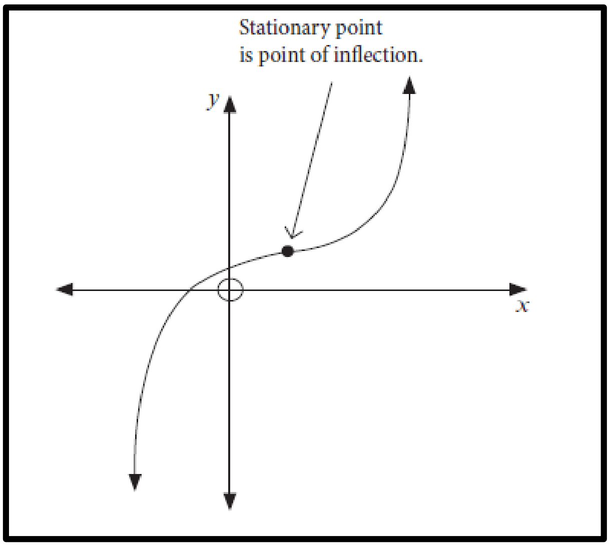
The derivative $f'(x)$ gives us the slope of a graph at any point, to find the coordinates of the turning points of a function $f(x)$, we solve $f'(x) = 0$.

To find the coordinates of the point of inflection, find the *derivative of the derivative*, $f''(x)$. This is called the second derivative. Solve for $f''(x) = 0$.



STEPS HOW TO DRAW THE CUBIC GRAPH

1. Find the y - intercept by finding $f(0)$. When $x = 0$, what is the value of y?
2. Find the x – intercepts by finding x value(s) where $f(x) = 0$. Factorise $f(x)$ to be able to work out these values, identify one factor by using the FACTOR THEOREM (trial and error) and Synthetic method to factorise completely.

   If $f(k) = 0$, then $x - k$ is a factor of $f(k)$
3. Find the stationary point or turning points by solving $f'(x) = 0$

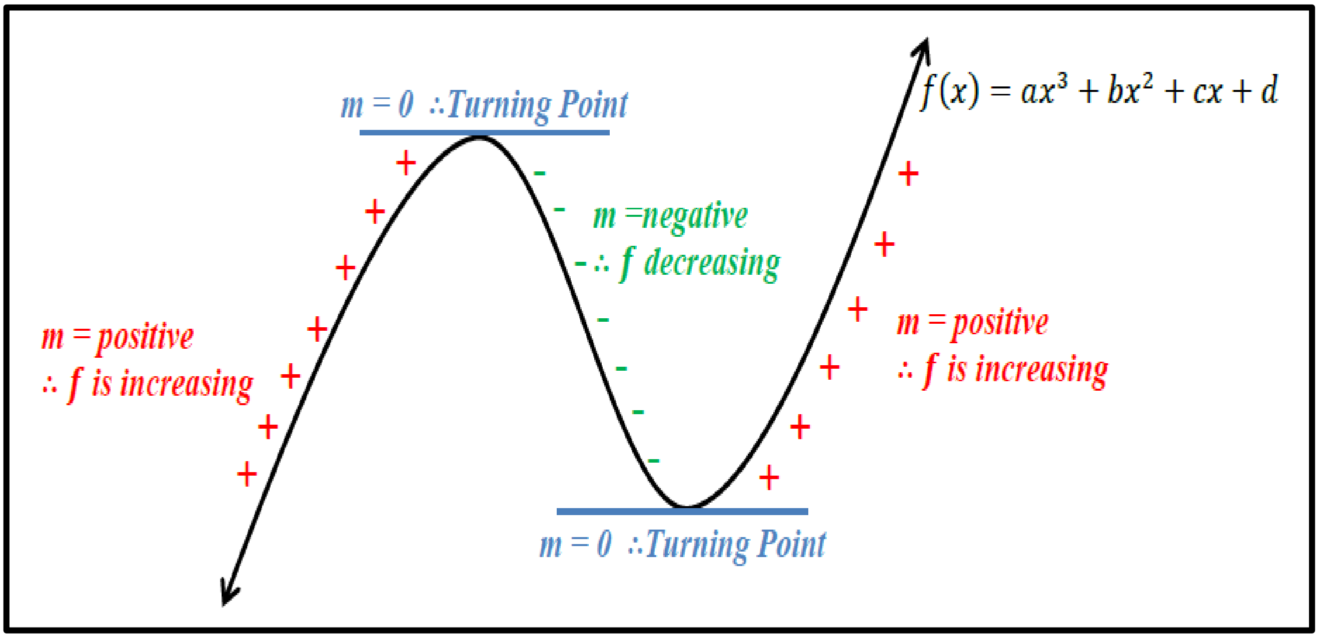
   NOTE: the three steps indicated above are very important. Sketched graph should have all the above points with correct identification of the shape explained below.

4. Determine the coordinates of the Point Of Inflection (make $f''(x) = 0$)
5. The shape of the graph if:

$a > 0$	**POSITIVE**	
$a < 0$	**NEGATIVE**	

THE SECOND DERIVATIVE (CONCAVITY)

Concavity refers to the shape of the graph. If the gradient of the curve is increasing as x increases, then we say tha graph is concave up, if the gradient gradient is decreasing, we say the graph is concave down.



A parabola is always concave up or concave down. However, the graph of a third degree polynomial has a point where the concavity changes. We call this point the point of inflection.

If $a > 0$, the concavity changes from being concave down to concave up at the point of inflection.

If $a < 0$, then the concavity changes from being concave up to concave down at the point of inflection.

WORKED EXAMPLE 1:

Sketch the graph of $f(x) = x^3 - 4x^2 - 11x + 30$

STEP 1: Find the y - intercept by finding $f(0)$. When $x = 0$, what is the value of y?

$y = f(0) = (0)^3 - 4(0)^2 - 11(0) + 30$

$y = 30$

So, the y – intercept is at $(0; 30)$

STEP 2: Find the x – intercepts by finding x value(s) where $f(x) = 0$. Factorise $f(x)$ to be able to work out these values, identify one factor by using the FACTOR THEOREM (trial and error) and Synthetic method to factorise completely.

$f(1) = (1)^3 - 4(1)^2 - 11(1) + 30 = 16$

So, $(x - 1)$ is NOT a factor.

$f(-1) = (-1)^3 - 4(-1)^2 - 11(-1) + 30 = 36$

So, $(x + 1)$ is NOT a factor.

$f(2) = (2)^3 - 4(2)^2 - 11(2) + 30 = 0$

So, $(x - 2)$ is a factor.

Now, use the synthetic method, this method is very quick and easy to use once you can use it accurately.

$$\begin{array}{r|l} & \mathbf{1 - 2 - 15 + 0} \\ \hline \mathbf{2} & \mathbf{1 - 4 - 11 + 30} \\ & \quad \mathbf{+2 \; - \; 4 - 30} \end{array}$$

$$a = 1\,; b = -2\,; c = -15$$

$$\therefore 1x^2 - 2x - 15$$

$\therefore x^3 - 4x^2 - 11x + 30 = (x - 2)(x^2 - 2x - 15)$

Factorise the trinomial

$\therefore x^3 - 4x^2 - 11x + 30 = (x-2)(x-5)(x+3)$

So when $y = 0$;

$x - 2 = 0$ $x = 2$	or	$x - 5 = 0$ $x = 5$	or	$x + 3 = 0$ $x = -3$

x-intercepts are at:

$(2; 0)$; $(5; 0)$ or $(-3; 0)$

STEP 3: Find the stationary point or turning points by solving $f'(x) = 0$

$f(x) = x^3 - 4x^2 - 11x + 30$

$f'(x) = 3x^2 - 8x - 11$

When $f'(x) = 0$, then $3x^2 - 8x - 11 = 0$

$\therefore (x+1)(3x-11) = 0$

$x = -1$ or $x = \frac{11}{3} = 3{,}7$

Now substitute the x - values into the **ORIGINAL** equation to determine the corresponding y - values.

$\therefore y = f(-1) = (-1)^3 - 4(-1)^2 - 11(-1) + 30$

$\therefore y = -1 - 4 + 11 + 30$

$\therefore y = 36$

MAXIMUM TURNING POINT: $(\mathbf{-1; 36})$

$$\therefore y = f\left(\frac{11}{3}\right) = \left(\frac{11}{3}\right)^3 - 4\left(\frac{11}{3}\right)^2 - 11\left(\frac{11}{3}\right) + 30$$

$\therefore y = -14{,}81$

MINIMUM TURNING POINT: $(\mathbf{\frac{11}{3}; -14,81})$

STEP 4: Determine the coordinates of the Point Of Inflection (make $f''(x) = 0$)

$f'(x) = 3x^2 - 8x - 11$

$f''(x) = 6x - 8$

$0 = 6x - 8$

Where:

$$x = \frac{8}{6} = \frac{4}{3} = 1{,}3$$

So point of inflection is at $x = \frac{4}{3}$

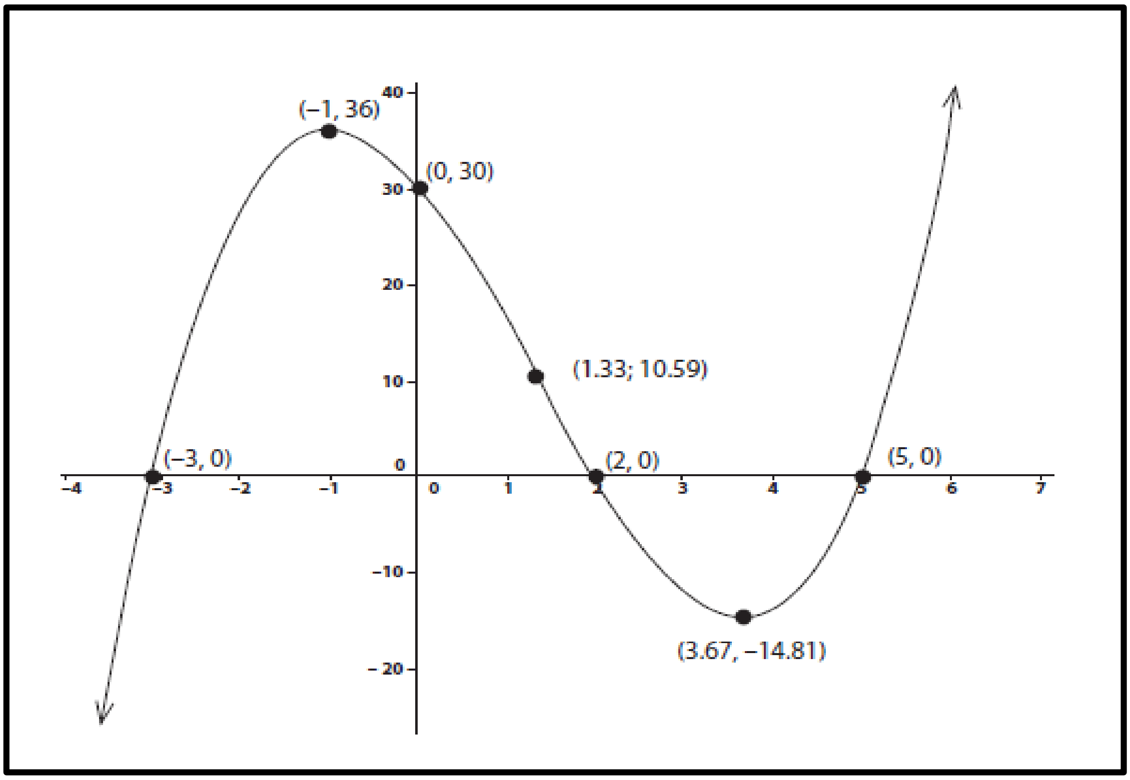
Now substitute the x - value into the **ORIGINAL** equation to determine the corresponding y - values.

$$\therefore y = f\left(\frac{4}{3}\right) = \left(\frac{4}{3}\right)^3 - 4\left(\frac{4}{3}\right)^2 - 11\left(\frac{4}{3}\right) + 30$$

$\therefore y = 10{,}59$

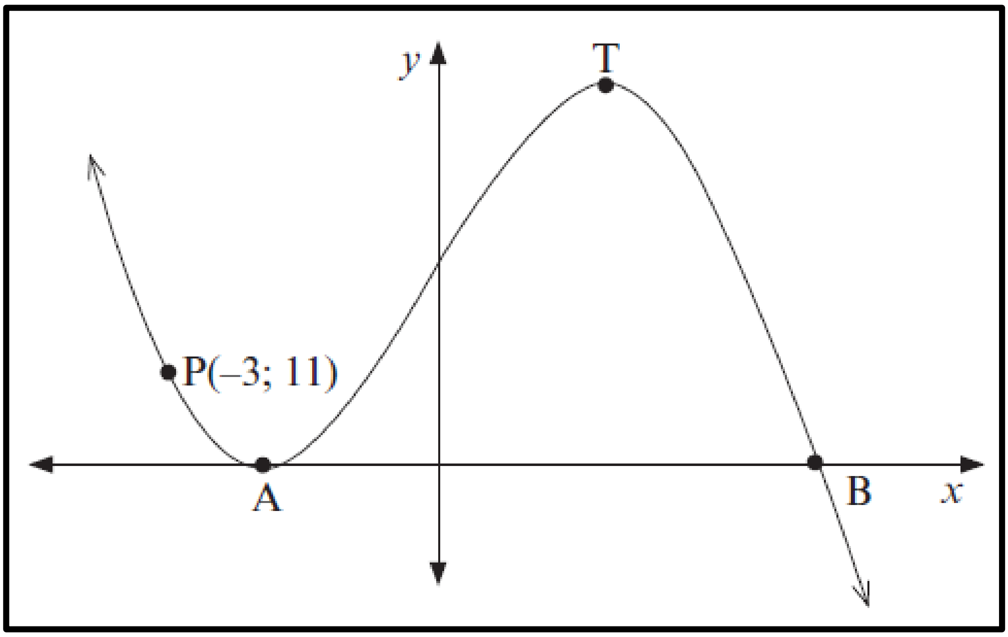
POI: $\left(\frac{4}{3}; \mathbf{10{,}59}\right)$

STEP 5: Sketch the graph of $f(x) = x^3 - 4x^2 - 11x + 30$, with all the point clearly identified on the graph.



WORKED EXAMPLE 2:

Sketched below is the graph of $g(x) = -2x^3 - 3x^2 + 12x + 20 = -(2x-5)(x+2)^2$. A and T are turning points of g. A and B are the x-intercepts of g. $P(-3; 11)$ is a point on the graph.



a) Determine the length of AB.

Since A and B are the x-intercepts of g they are the solutions of $-(2x-5)(x+2)^2 = 0$

So:

$-(2x-5) = 0$	$(x+2)^2 = 0$
Divide by -1	Square root both sides
$2x - 5 = 0$	$x + 2 = 0$
$\therefore x = \frac{5}{2} = 2{,}5$	$\therefore x = -2$

The distance between $x = -2$ and $x = \frac{5}{2} = 2{,}5$ is 4,5 units

b) Determine the x-intercept of T.

T is a turning point, so, $g'(x) = 0$

$g(x) = -2x^3 - 3x^2 + 12x + 20$

$g'(x) = -6x^2 - 6x + 12 = 0$

$-6(x^2 + x - 2) = 0$

Divide by -6

$\therefore x^2 + x - 2 = 0$

$\therefore (x + 2)(x - 1) = 0$

$\therefore x = -2$ or $x = 1$

So, the x-coordinate of T is 1

c) Determine the equation of the tangent to g at $P(-3; 11)$ in the form $y =$..

$m = g'(x) = -6x^2 - 6x + 12$

$m = g'(-3) = -6(-3)^2 - 6(-3) + 12$

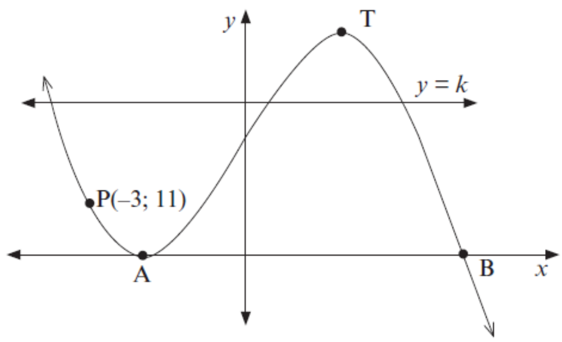
$m = -24$

So the equation of the tangent line is:

$y - 11 = -24(x + 3)$

$y = -24x - 61$

d) Determine the value(s) of k for which $-2x^3 - 3x^2 + 12x + 20 = k$ has three distinct roots.

The graph of $y = k$ is shown together with $g(x)$ below. Using these graphs we can observe that, provided the line lies above the y-value of A and below that of T, the equation $-2x^3 - 3x^2 + 12x + 20 = k$ will have three distinct roots. At T, $y = g(1) = -2 - 3 + 12 + 20 = 27$. So for $0 < k < 27$ the equation has 3 distinct roots.	

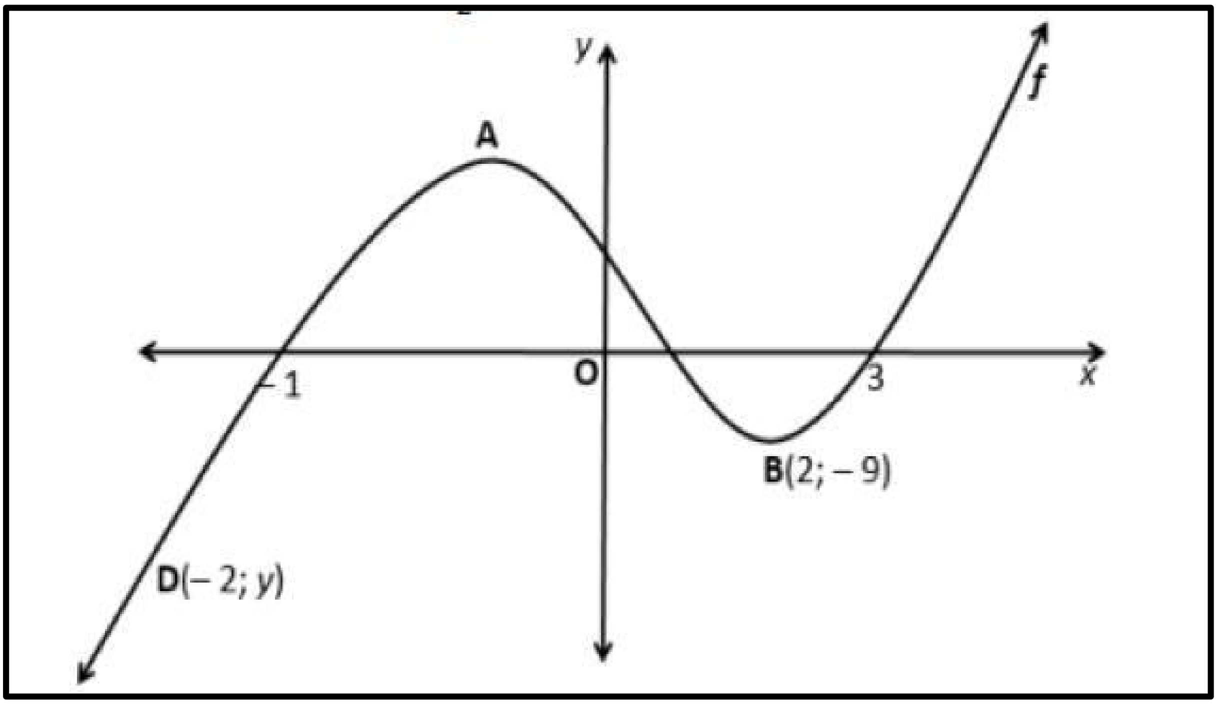
e) Determine the x-coordinate of the point of inflection.

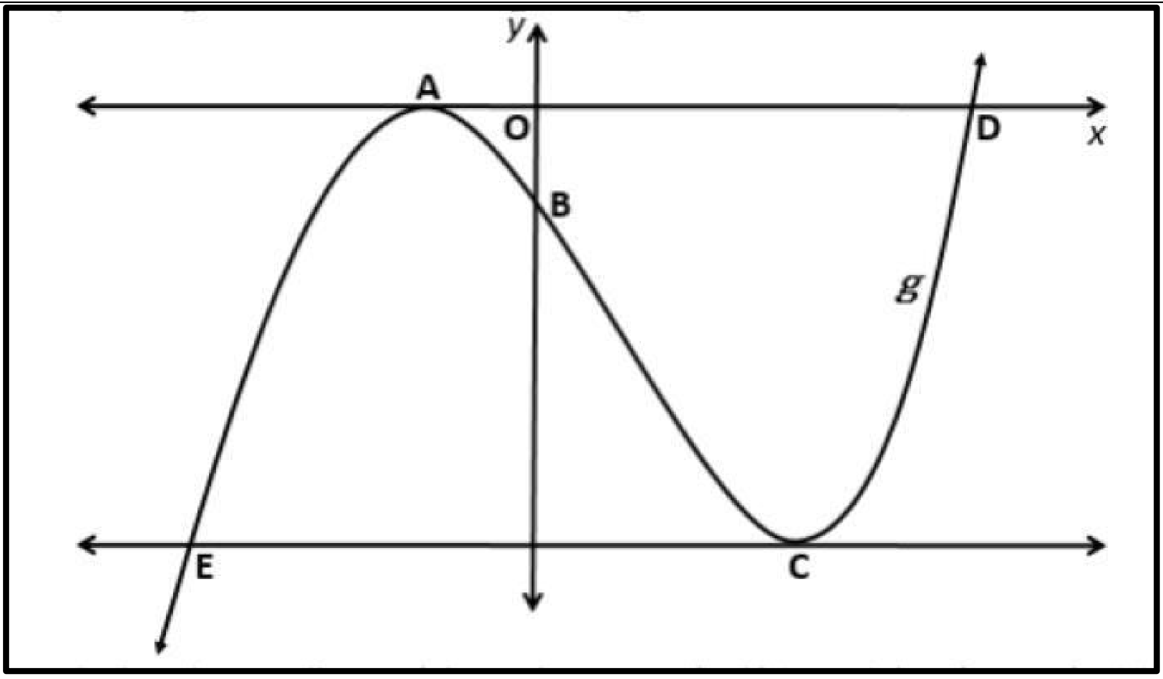
$$g''(x) = -12x - 6$$
$$-12x - 6 = 0$$
$$x = -\frac{6}{12} = -\frac{1}{2}$$

HOMEWORK ACTIVITY

1.	Given: $f(x) = x^3 - 5x^2 - 8x + 12$
1.1	Calculate the y-intercept of the graph of f.
1.2	Calculate the x-intercepts of the graph of f.
1.3	Determine the coordinates of the turning points.
1.4	Find the coordinates of the point of inflection of f.
1.5	Draw a neat sketch of f. Clearly show all intercepts and stationary points.
1.6	Determine the
1.6.1	Gradient of the curve at $x = -3$
1.6.2	Equation of the tangent to the curve at $x = -3$
1.7	For which value(s) of x will:
1.7.1	$f(x) > 0$?
1.7.2	The function be negative? $f(x) < 0$
1.7.3	$f(x) = 0$?
1.7.4	The gradient be negative?
1.7.5	$f'(x) \geq 0$?

1.7.6	The gradient be zero?
1.7.7	$f(x)$ be concave up?
1.7.8	$f(x)$ be concave down?
2	Given: $g(x) = -x^3 - 4x^2 + 7x + 10$
2.1	Calculate the y-intercept of the graph of f.
2.2	Show that 2 is the root of g. Hence solve for $g(x) = 0$
2.3	Determine the coordinates of the turning points.
2.4	Draw a neat sketch of f. Clearly show all intercepts and stationary points.
2.5	Find the coordinates of the point of inflection of f.
2.6	Determine the
2.6.1	Gradient of the curve at $x = 1$
2.6.2	Equation of the tangent to the curve at $x = 1$
2.7	For which value(s) of x will:
2.7.1	The function be positive?
2.7.2	$g(x) \leq 0$?
2.7.3	The function be zero?
2.7.4	$g'(x)$ be negative?
2.7.5	The gradient be positive?
2.7.6	The gradient be zero?
2.7.7	$g(x)$ be concave up?

2.7.8	$g(x)$ be concave down?
3	In the diagram below, the graph of $f(x) = 2x^3 + bx^2 + cx + d$ is drawn. The x-intercepts of f are: $-1; \frac{1}{2}$ and 3. A and $B(2; -9)$ are turning points of f. 
3.1	Find the equation of f.
3.2	Calculate the coordinates of A if it is given that: $f(x) = 2x^3 - 5x^2 - 4x + 3$.
3.3	Determine the equation of the tangent to f, through the point D, in the form $y = mx + c$.
4	The sketch below represents cubic function $g(x) = x^3 - 3x^2 - 9x - 5$. A and C are local maximum and minimum turning points. D is a x-intercept of g. B is the y-intercepts of g. E is a point on g such that EC is a tangent to g at C.



4.1	Calculate the coordinate of the x-intercept of g if $(x + 1)$ is a factor of $g(x)$.
4.2	Determine the coordinates of A and C the turning points of g.
4.3	Determine the x-coordinate of E.
4.4	Determine:
4.4.1	The coordinate of the point of inflection of g.
4.4.2	The equation of the tangent to g at the point of inflection.
5	The gradient of a tangent to the curve $f(x) = ax^3 + bx^2$ at point $C(1; 7)$ is 17.
5.1	Calculate the values of a and b.
5.2	If it is given that $a = 3$ and $b = 4$, determine the coordinates of one other point on the curve where the gradient of the curve is also equal to 17.
5.3	Sketch the graph of $f(x) = 3x^3 + 4x^2$ is concave up.